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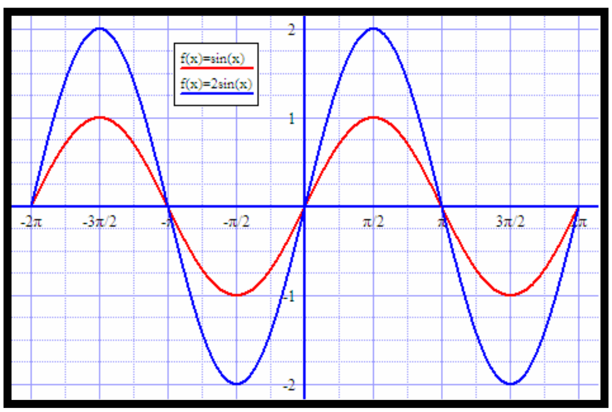
Investigating the Sine Curve

This report investigates the sine curve in the form o $y = a\,sin[b(x-c)] + d$ f, and how that relates to the graph of the sine curve. In particular, it would be investigated how the different variables (a, b, c and d) effect the way that the graph is drawn and then seeing if the rule can be generalized to apply to any form of the equation.

The first thing to do would be to allow b, c and d to be 0, which would mean that the equation $y = a\,sin[b(x-c)] + d$ takes the form of: $y = a\sin(x)$. It can be seen from Graph 1.1 that when a is 1 what graph you get (the red graph) and when a is allowed to be 2 what the graph looks like (the blue graph). From the graph below it can be seemn that increasing a stretches the graph by the factor of change in a. In simple words, the graph of $y = 2\sin(x)$ would be twice the height of $y = \sin(x)$ as is clearly seen from Graph 1.1.

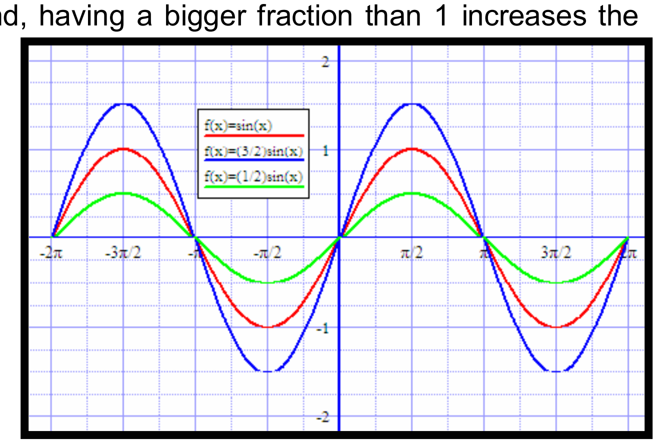


Graph 1.1: Graph of **y = $\sin(x)$** and

When I change the value for a, all the value of the sine curve get multiplied by that value of a, which is 2 in this case. By multiplying all the values of the curve by a the height of each point in the curve increases while there is no change in the position of the graph in the x-axis. For the original sine graph ($y = \sin(x)$) it is common knowledge that the relative minima and maxima are -1 and 1 respectively, however when the a is changed the original minima and maxima are also multiplied by a and therefore the new minima and maxima would become $-a$ and a respectively. The rest of the graph also gets stretched by a as this is the number that the whole curve is being multiplied by.

This time when we change the value of a we would change it into a fraction, one fraction would be smaller than 1 whereas the other one would bigger than 1. It can be seen that when you have a fraction that is smaller than 1 for your a value the height of the sine curve ill decrease to the height of a. On the other hand, having a bigger fraction than 1 increases the height of the sine curve to the a value as can be seen from Graph 1.2.

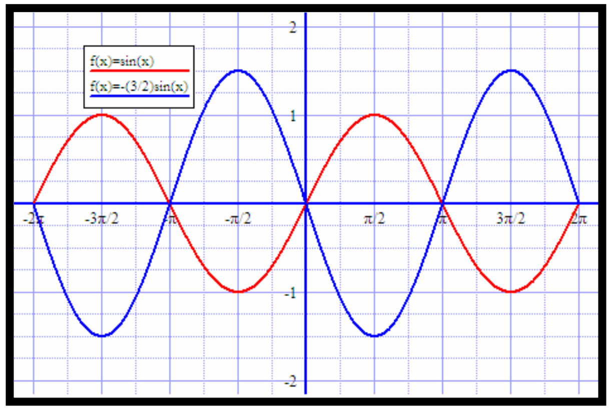
From these two graphs, it can be said that when a is bigger than 1 the graph stretches outwards, whereas when a is smaller than one the graph will stretch inwards. It is also seen that changing the a of the sine curve changes



Graph 1.2: Graph of **y = $\sin(x)$**, $y = \frac{3}{2}\sin(x)$ and

the minima and maxima of the graph to $\pm a$. All the values of the sine curve are being multiplied by a and therefore there is only a vertical shift and no horizontal shift.

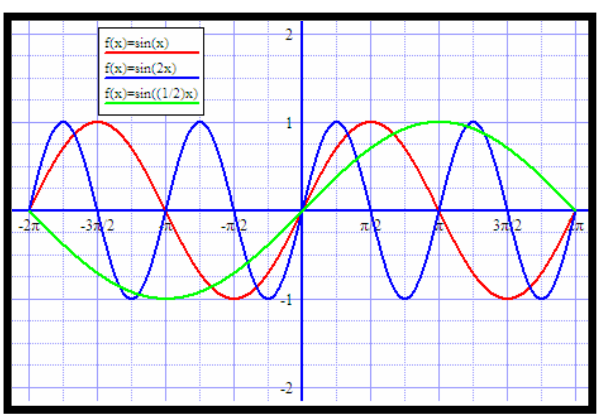
We must not forget that a could be a negative number as well and this has been explored with Graph 1.3 on the right hand side. In the sine curve $y = a\sin(x)$ if the a becomes negative then all the values of the sine curve would be multiplied by the negative number. This would mean that all the positive values become negative and all the negative values become positive. To put this into simple terms the graph flips over with respect to the x-axis. Other than flipping over with respect to the x-axis, the graph stretches according to a. In Graph 1.3 $y = \sin(x)$ has been flipped over and stretched by a factor of $-\frac{4}{3}$ to give the new graph of $y = -\frac{4}{3}\sin(x)$.



Graph 1.3: Graph of **y = sin** (x) and

Overall, it can be seen that in $y = a\,sin[b(x-c)] + d$ the a represents the 'height' of the graph. In mathematics this 'height' is known as the amplitude of the graph, meaning that in the equation above, a represents the amplitude of the graph. When a is negative the graph is not only stretched by a but it is also flipped over the x-axis. This is happening because all the y values of the graph are multiplied by a which causes there to be a shift in the amplitude of the graph without affecting the horizontal position of the graph.

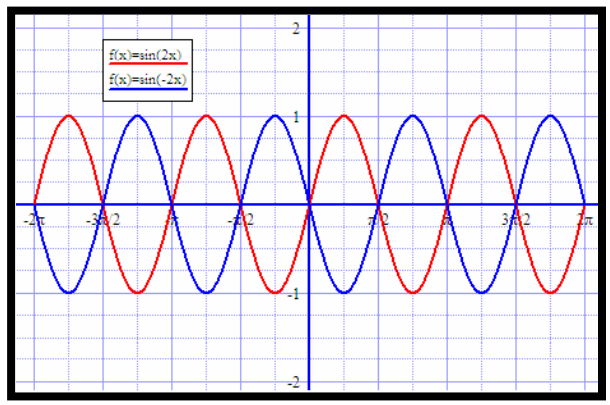
Moving on, we now set a, c and d at 0 giving the equation $y = a\,sin[b(x-c)] + d$ the form of $y = \sin(bx)$. It is observed that changing the b has the same effect as changing the a in the equation, except rather than the curve stretching vertically it stretches horizontally. This is known as a period in mathematics. It is the interval between likewise values, or in simpler terms the interval before the graph 'repeats' itself. The graph below shows the curve stretching inwards or outwards (depending on whether the b value is greater than or smaller than 1) without changing the vertical position of the curve.



Graph 2.1: Graph of **y =** $\sin(x)$, $y = \sin(2x)$ and $y = \sin(\frac{1}{2}x)$

The reason that this horizontal stretch occurs is because all the x values of the curve are multiplied by a factor of b. This means that a lower value of x is required to get the same value as the original graph. An example of this can be seen when $x = \frac{\pi}{2}$ in the original graph, $y = \sin\frac{\pi}{2} = 1$ (as can be seen from Graph 2.1). However it can be seen that when b is 2, a lower value of x is required to get $y = 1$. When $y = \sin(2x)$ the value of x would need to be $\frac{\pi}{4}$ so that $y = \sin\left(2\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$. This tells us that all the values of x would be half of what they were originally causing the curve to become narrower and therefore decreasing the period of the curve. On the other hand, when the value of b is smaller than 1 a bigger value of x would be required to get the same value of y. For example, if you need the graph $y = \sin(\frac{1}{2}x)$ to give you the same value as the graph of $y = \sin(x)$ when x is $\frac{\pi}{2}$ then the value of x would need to be π in the curve $y = \sin(\frac{1}{2}x)$.

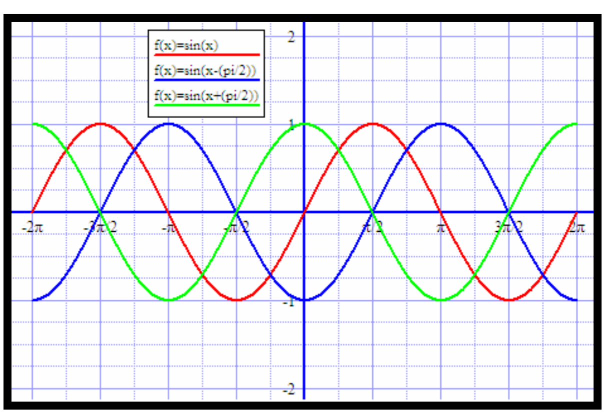
As it has already been mentioned, by changing the b of the equation you either stretch the graph inwards or outwards (depending if the value of b is bigger or smaller than 1). When you increase the value of b you decrease the period of the graph and when b is decreased the period of the graph increases. However, what happens if the value of b is a negative number? When the value of b is negative it has the same effect as when a is negative. This is because $y = \sin(-x)$ is the same as $y = -\sin(x)$. This means that a negative b value would flip the graph with respect to the x- axis as shown in Graph 2.2.



Graph 2.2: Graph of $y = \sin(2x)$ and

In general it is observed that in the equation $y = a\sin[b(x - c)] + d$ the period of the curve is given by $\frac{2\pi}{b}$. By changing the value of b you can change the period of the graph. When you increase the b the period of the graph decreases and when you decrease the b the period of the graph increases. This means that the curve only stretches in the horizontal direction while there is no stretch in the vertical direction of the curve.

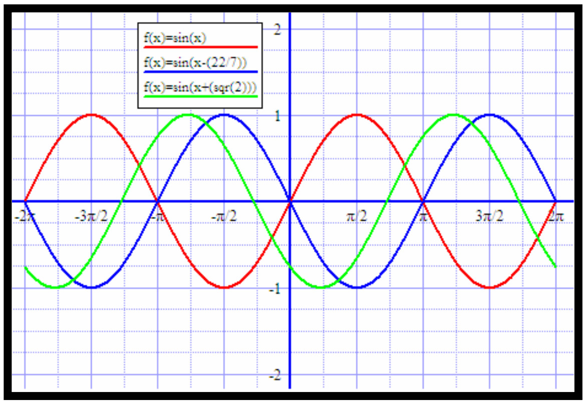
When you make a, b and d 0 you get the equation $y = \sin(x - c)$. In this equation, c corresponds to the horizontal translation of the graph. When c is a positive number the translation is a horizontal translation to the right, whereas when the value of c is negative the graph translates to the left by c units as illustrated by Graph 3.1.



Graph 3.1: Graph of $y = \sin(x)$, $y = \sin(x - \frac{\pi}{2})$ and

When you change the value of c you subtract the original value of x by c, and therefore the x

value of the new curve would be increased by c to match the y value. For example, when $x = \pi$, where c is 0, $y = \sin(\pi) = 0$. However when there is a value for c,$y = \sin(x - c)$ cannot be 0 unless $x - c = \pi$ in which case x must be equal to $\pi + c$. This would translate the point and the rest of the graph by c units to the right. This would only cause a translation meaning that there is no change in the amplitude or the period of the curve. From Graph 3.1, it can also be seen that that when c is a negative number, the equation becomes $y = \sin(x + c)$ and this causes the graph to translate c units to the left.



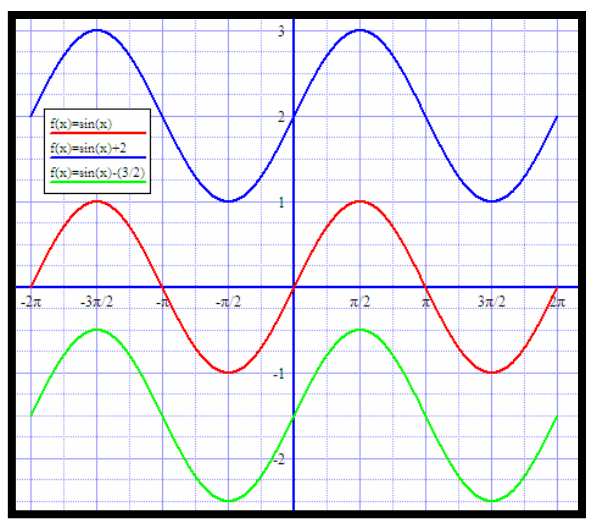
Graph 3.2: Graph of $y = \sin(x)$, $y = \sin\left(x - \frac{22}{7}\right)$ and $y = \sin(x + \sqrt{2})$

From Graph 3.2 it can be seen that the value of c can be any number. It can either be a fraction, a whole number or even an irrational number. The same concept follows for fractions and irrational numbers as well, where the value of c dictates the horizontal translation of the curve.

Overall, it is observed from this graph and the above graph that the c in the equation $y = a\sin[b(x - c)] + d$ corresponds to the horizontal position of the sine graph. When you change the value of c the sine graph would be horizontally translated to the right by c units if the value of c is positive and likewise to the left when the value of c is negative.

Now we will set a, b and c 0 so that we obtain the equation $y = \sin(x) + d$. Changing the value of d has a similar affect as changing the value of c, however the difference is that instead of there being a horizontal translation there is a vertical translation. Graph 4.1 is used to describe this translation, and it is seen from that graph that when d is positive there is a vertical shift upwards in the sine curve and when the value of d is negative the sine curve shits down vertically.

The value of d is the value that the y values are added by. Therefore, by changing the value of d to a positive value you ensure that the graph will be translated upwards by d units as the d value is added to the original y value. An example of this is that in the original graph $y = \sin(x)$, when $x = \pi$, the value of y is 0. However if you add a value of d to the equation so that it now looks like: $y = \sin(x) + d$ then the new value of y would be different. Let the value of d be 2 in the new curve when $x = \pi$ and your new value of y would be $y = \sin(\pi) + 2 = 0 + 2 = 2$ rather than 0 and



Graph 4.1: Graph of $y = \sin(x)$, $y = \sin(x) + 2$ and $y = \sin(x) - \frac{3}{2}$

therefore translating the graph upwards.

On the other hand, when the value of d is a negative number the graph shifts downwards. An example of this can be seen from Graph 4.1. In the equation $y = \sin(x) + d$, if x was π and d was $-\frac{3}{2}$ then y value would be $-\frac{3}{2}$ rather than 0. For this to happen the curve must shift downwards and therefore a negative value of d means that the sine curve shifts down vertically.

In general it can be said that the value of d corresponds to the vertical position of the sine curve. This means that if you change the d in the equation $y = a\sin[b(x-c)] + d$ then you can translate the curve in a vertical direction by d units.

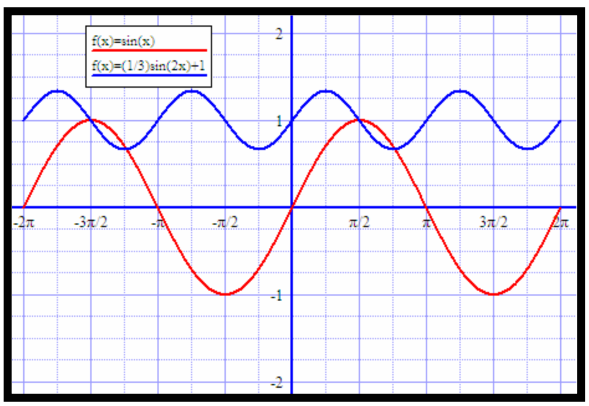
All in all we can talk about how the different variables of the equation $y = a\sin[b(x-c)] + d$ effect the sine curve and what they all represent. The first thing that we found out was that the variable a represents the amplitude of the graph meaning the height of the graph. The amplitude of the graph would be the value of a. On the other hand, the value of b corresponds to the period of the curve where the period is given by $\frac{2\pi}{b}$. Moving on, we saw that the variables c and d correspond to the horizontal and vertical translation of the sine curve respectively. The curve is translated c to the right horizontally and d units vertically upwards from the original sine curve.

Since we know the general rule of the variables, it can help predict what the graphs would look like, and how they would have been translated from the original curve. This can be done by just identifying the different variables a, b, c and d. We will predict the graphs of $y = \frac{1}{3}\sin(2x) + 1$, $y = -3\sin(x - \frac{\pi}{4})$ and $y = 4\sin\left[2\left(x + \frac{\pi}{6}\right)\right] - \frac{3}{2}$ and then we will test our predictions by graphing these equations.

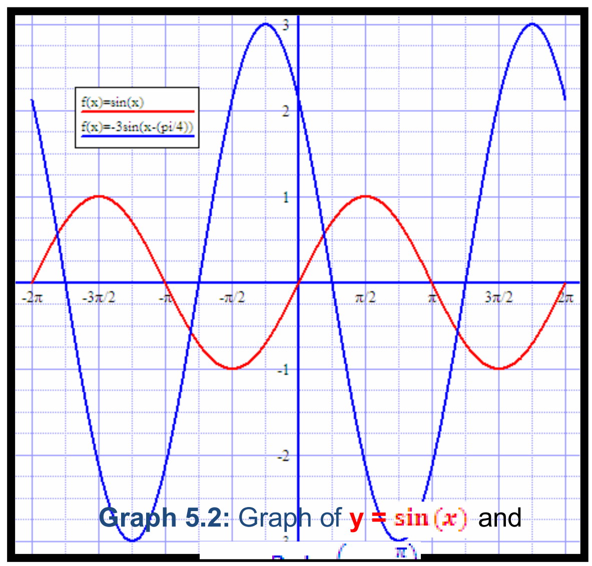
First of all we examine the equation $y = \frac{1}{3}\sin(2x) + 1$ and identify the different variables. In this equation $a = \frac{1}{3}, b = 2, c = 0$ and $d = 1$. This means that the amplitude of the graph would be $\frac{1}{3}$, causing the original curve to be stretched inwards, whereas the period of the new curve would be $\frac{2\pi}{2} = \pi$. Since the value of c is 0, there would be no translation in the horizontal direction, however the curve would be translated upwards in the vertical direction by 1 unit. On the other hand, the second equation $y = -3\sin(x - \frac{\pi}{4})$ would be look different from the first equation. In this equation the variables are as following, $a = -3, b = 0, c = \frac{\pi}{4}$ and $d = 0$. This means that the amplitude of this graph would be 3, an outwards stretch from the original graph. The negative means that the graph would be flipped over with respect to the x- axis. Since the b is 0 there would be change in the period of the graph, whereas the graph would be translated by $\frac{\pi}{4}$ units to the right horizontally with no change in the vertical position of the graph since $d = 0$.

We can now examine the last equation $y = 4\sin\left[2\left(x + \frac{\pi}{6}\right)\right] - \frac{3}{2}$. In this equation the variables look as following: $a = 4, b = 2, c = -\frac{\pi}{6}$ and $d = -\frac{3}{2}$. This means that the amplitude of this new graph would be 4, causing an outwards stretch of the original graph. Since $b = 2$ the period of the graph would be reduced to $\frac{2\pi}{2} = \pi$. Since $, c = -\frac{\pi}{6}$ the graph would be translated $\frac{\pi}{6}$ units to the left horizontally whereas the curve would shift downwards vertically by $-\frac{3}{2}$ units.

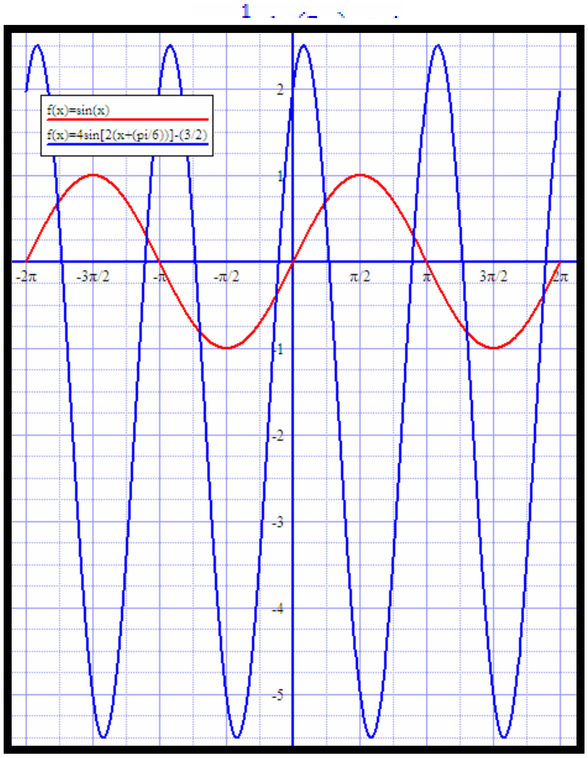
The graphs below indicate that the predications that were made about the translation of the graphs hold true, and therefore this means that the conclusions we had come up with about the variables were indeed correct.



Graph 5.1: Graph of **$y = \sin(x)$** and



Graph 5.2: Graph of **$y = \sin(x)$** and



Graph 5.3: Graph of $y = \sin(x)$ and $y = 4\sin\left[2\left(x + \frac{\pi}{6}\right)\right] - \frac{3}{2}$

So far in this investigation it has been observed that if we are given an equation which is in the form $y = a\sin[b(x - c)] + d$ we can predict the shape and position of the sine graph. However what if the equation is not in that form as happens a lot of times? If the equation is not written in the form as written above then it can be transformed to that form as has been done with three examples below. The three equations that would be transformed are: $y = 2\sin\left(\frac{3x-\pi}{2}\right) + \frac{5}{2}$, $y = \frac{2\sin\left(2x - \frac{2\pi}{3}\right)}{3}$ and lastly the graph $y = -\sin(\pi x - \pi) + \frac{\pi}{2}$.

If we look at the first equation, then the first thing to do in that equation would be to split the fractions so that the equation now looks like: $y = 2\sin\left(\frac{3x}{2} - \frac{\pi}{2}\right) + \frac{5}{2}$. Once that has been done we can now factorize the brackets' section by $\frac{1}{2}$, and therefore the equation would look like: $y = 2\sin\left[\frac{1}{2}(3x - \pi)\right] + \frac{5}{2}$. This final step would now be to factorize the entire equation by 3 so that the equation looks like: $y = 2\sin\left[\frac{3}{2}\left(x - \frac{\pi}{3}\right)\right] + \frac{5}{2}$. Now that this equation is in the form that we require it to be, we can see the difference variables and see how the sine curve has been transformed. In this graph $a = 2$, $b = \frac{3}{2}$, $c = \frac{\pi}{3}$ and $d = \frac{5}{2}$. This would suggest that the graph as an amplitude of 2, causing the graph to stretch outwards by 2 units. The period would be $\frac{2\pi}{\frac{3}{2}}$ making the period be $\frac{4\pi}{3}$. This c and d value tells us that graph has been translated horizontally by $\frac{\pi}{3}$ units to the right whereas it has been translated $\frac{5}{2}$ units vertically upwards.

We can use a similar method for the rest of the equations:

$$y = \frac{2\sin\left(2x - \frac{2\pi}{3}\right)}{3} \qquad \text{splitting the fraction}$$

$$y = \frac{2}{3}\sin\left(2x - \frac{2\pi}{3}\right) \qquad \text{factorizing by 2}$$

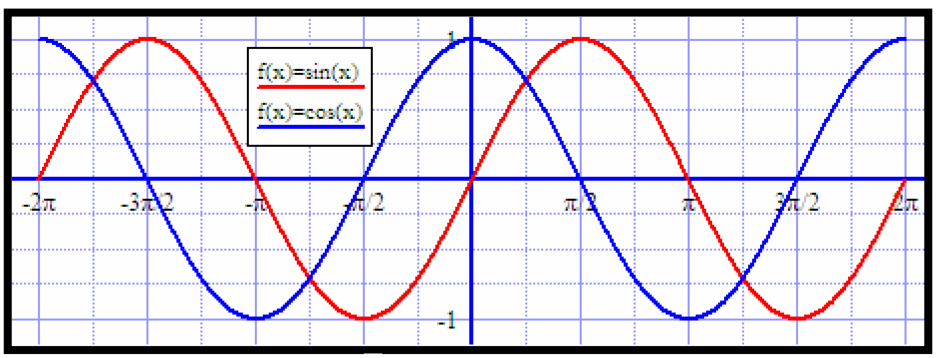
$$y = \frac{2}{3}\sin\left[2\left(x - \frac{\pi}{3}\right)\right]$$

From this it can be seen that $a = \frac{2}{3}$, $b = 2$, $c = \frac{\pi}{3}$ and $d = 0$. This would mean that the amplitude of the graph would be $\frac{2}{3}$, causing the original graph to be stretched inwards by $\frac{2}{3}$ units. Not only this, by the period of the new graph would be π and there would be a horizontal translation in the graph of $\frac{\pi}{3}$ units to the right hand side while there is no translation in the vertical translation of the graph.

Lastly we will look at transforming the last equation into the desired form. The first step, and only step to transform the equation $y = -\sin(\pi x - \pi) + \frac{\pi}{2}$ is to factorize by π, as this would change the equation to: $y = -\sin[\pi(x-1)] + \frac{\pi}{2}$. Now it can be seen that the value of $a = -1, b = \pi, c = 1$ and lastly, $d = \frac{\pi}{2}$. This means that the new curve would be flipped over with respect to the x- axis, however there will be no change in the amplitude. The period of the curve would be given by $\frac{2\pi}{\pi} = 2$. And finally it is observed the graph translates 1 unit horizontally to the right whereas the graph translates upwards vertically.

From all the graphs, discussions and examples it is clearly seen that when the sine curve is written in the form $y = a\sin[b(x-c)] + d$ we can get a lot about the graph. The represents the amplitude of the curve, the period of the graph is written by $\frac{2\pi}{b}$ whereas the values of c and d translate the graph horizontally and vertically respectively.

So far in this investigation we have been gathering the shape and look of the graph from just the equation, however we will now do it the other way around. We will find the equation of the curve from the graph.



Graph 6.1: Graph of $y = \sin(x)$ and $y = \cos(x)$

The graph on the right hand side shows the sine curve and the cosine curve together. It can be observed that the curve of $y = \cos(x)$ looks just like the curve of $y = \sin(x)$ and therefore could be a translation. Just like the sine curve, the amplitude (a) of the cosine curve is 1, whereas, the period (b) of the cosine curve is 2π (just like sine curve), meaning that the b value of the cosine curve would be 1. The only difference in the sine and cosine curve seems to be that the cosine curve has been translated to the left hand side by $\frac{\pi}{2}$ units, meaning that the value of $c = -\frac{\pi}{2}$ and the value of $d = 0$. When we plug the numbers into the equation of the sine curve $y = a\sin[b(x-c)] + d$ the equation that we get is the following: $y = \sin\left(x + \frac{\pi}{2}\right)$. Since this equation is for the cosine curve we can say that $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$. However, there is another way to express this translation. It can be said that the graph of $y = \sin(x)$ was flipped with respect to the x axis and then translated horizontally to the right by $\frac{\pi}{2}$ units. When this is the case a becomes -1, b and d remain 0 and c changes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The equation for this new translation would look like $\cos(x) = -\sin\left(x - \frac{\pi}{2}\right)$ and is something else which could be used for calculations in trigonometry.

In conclusion, it was observed that there is a strong relationship between the variables of an equation and how the graph looks. In terms of the sine curve, the variables represent the amplitude, the period and the translations (horizontal and vertical) of the curve. This allowed us to come up with a graph without having to plot it, and when we plotted the graph our guess was shown to be correct indicating that the idea of the variables was correct. Near the end of the investigation, we used these variables and the sine curve to come up with a relationship between the sine and cosine curve which would then help us in future trigonometry calculations. Just like that, we can deduce different relationships between different sorts of sine, and cosine functions to help us in future calculations.

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