Infinite Surds

Mathematics Portfolio SL Type 1

November 10, 2008

"A mathematician is a blind man in a dark room looking for a black cat which isn't there."

Although there is no significant application of infinite surds in our daily lives, infinite surds can be classified as a recursion which means that the function defined gets used in its own definition. See below for an example.

http://www.vdschot.nl/jurilog/images/metro.jpg

and this humorous definition of recursion: **Recursion**
See "Recursion".

http://en.wikipedia.org/wiki/Recursion

A surd is defined as an irrational number which can only be expressed with the root symbol and has an infinite number of non-recurring decimals.

The following expression is an example of an infinite surd: $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$

This surd can be considered as a sequence of terms a_n where:

$$a_1 = \sqrt{1+\sqrt{1}}$$

$$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}}$$

$$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}$$

$$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}$$

etc.

A pattern is evident where for every consecutive term in the sequence a new (+ $\sqrt{1}$) is added to the end of the previous expression under all the previous roots. The formula for a_{n+1} in terms of a_n is therefore:

$$\boxed{a_{n+1} = \sqrt{1+a_n}}$$

The first eleven terms of the sequence to eleven decimal places including a_0 are as follows:

$a_0 \approx 1$

$a_1 \approx 1.41421356237$

$a_2 \approx 1.55377397403$

$a_3 \approx 1.59805318248$

$a_4 \approx 1.61184775413$

$a_5 \approx 1.61612120651$

$a_6 \approx 1.61744279853$

$a_7 \approx 1.61785129061$

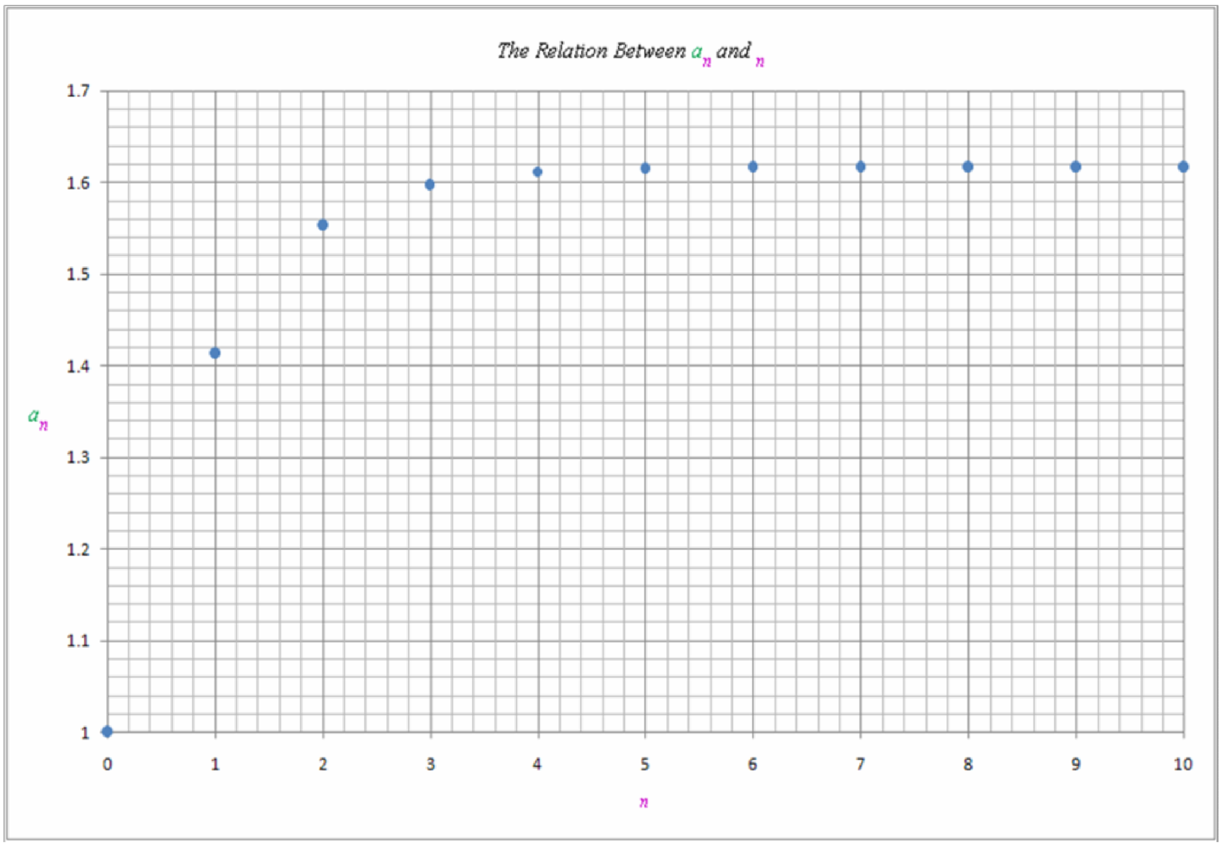
$a_8 \approx 1.61797753093$

$a_9 \approx 1.61801654223$

$a_{10} \approx 1.61802859747$

The first term in the sequence is actually $a_0 = \sqrt{1}$ since for every consecutive term a new (+ $\sqrt{1}$) is added on. Individual terms of an infinite surd are not necessarily surds themselves as $\sqrt{1}$ is equal to 1 which disqualifies it from being a surd.

Plotting the first eleven terms including a_0 on a graph reveals the behavior of the expression. The following graph illustrates the relation between n and a_n.



After the first five terms the data points increase by less and less suggesting an asymptote between 1.61 and 1.62. It is also clear that further points will not be much larger than a_{10} as the points already seem to be the same. Also, as the values get larger, the values of a_n and a_{n+1} will be almost the same with a difference that is too small to be significant.

This realization allows us to actually make them the same variable (x) in the previously deduced formula:

$$a_{n+1} = \sqrt{1 + a_n}.$$

$$\boxed{x = \sqrt{1+x}}$$

Giving us a new formula:

We can further simplify it:

$$(x = \sqrt{1+x})^2$$

By squaring the equation, the negative solution is removed because a negative multiplied by a negative gives a positive.

$$x^2 = 1 + x$$

$$x^2 - x - 1 = 0$$

The variable can be isolated from the rest of the equation with the use of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where: a = 1, b= -1, c= -1

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

By taking squaring the equation we got rid of the negative solution therefore the answer is:

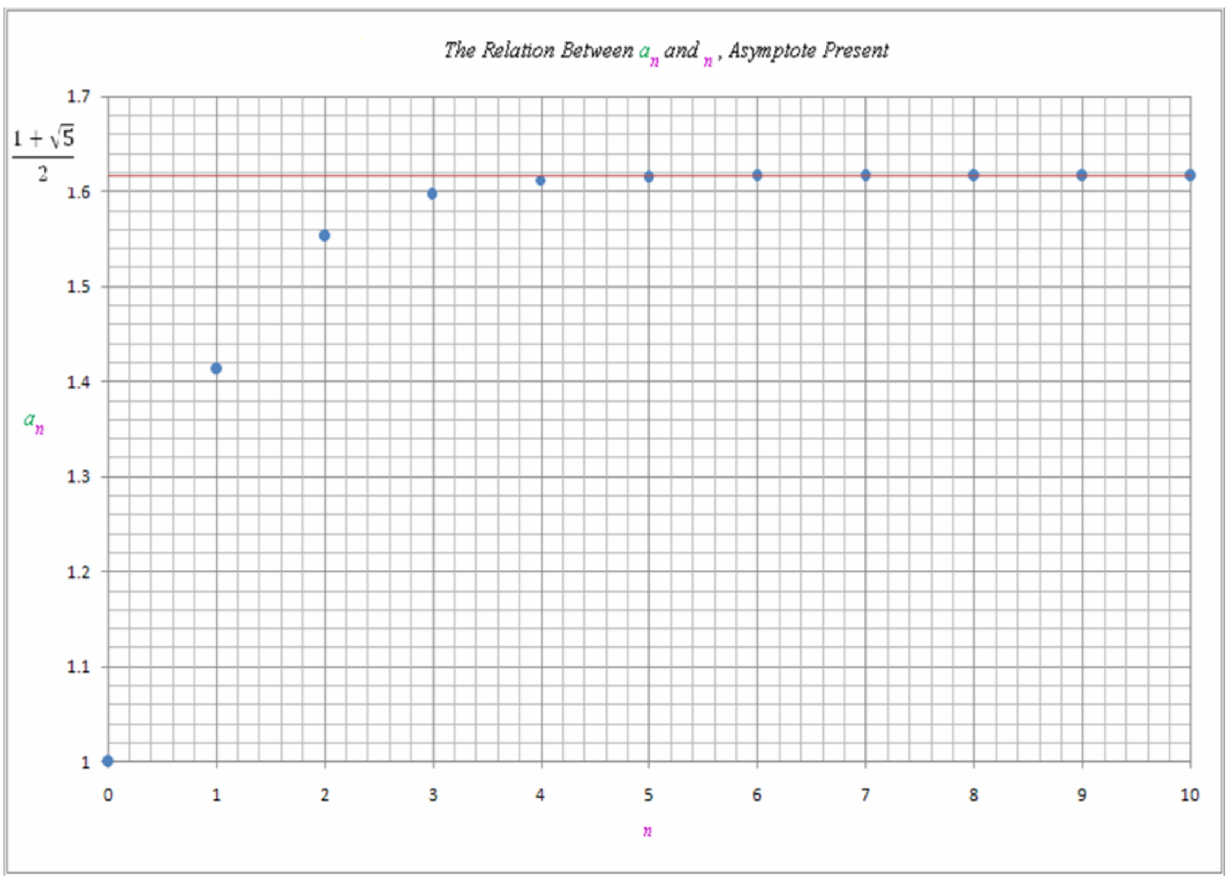
$$\boxed{x = \frac{1+\sqrt{5}}{2}}$$

Or in decimal form: $x \approx 1.61803398875$

The graph proves this as all the values are positive starting from the first term (a_0) and the trend suggests that they will remain so.

1.61803398875 appears to be the asymptote and falls between the originally predicted 1.61 and 1.62. The difference between the solution and a_{10} is only 0.00000539128 proving that further points are going to be almost the same as a_{10}.

By plotting the solution on the graph, we can see that the answer is where the asymptote would be.



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The same procedure will now be done with another surd to verify the method. The second surd is:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$$

This surd can once again be considered in terms of a_n where:

$$a_1 = \sqrt{2+\sqrt{2}}$$

$$a_2 = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$a_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$$

etc.

A pattern is once again evident where for every consecutive term in the sequence a new (+ $\sqrt{2}$) is added to the end of the previous expression under all the previous roots. The formula for a_{n+1} in terms of a_n is therefore:

$$\boxed{a_{n+1} = \sqrt{2+a_n}}$$

The first eleven terms of this sequence to eleven decimal places including a_0 are as follows:

$a_0 \approx 1.41421356237$

$a_1 \approx 1.84775906502$

$a_2 \approx 1.96157056081$

$a_3 \approx 1.99036945334$

$a_4 \approx 1.99759091241$

$a_5 \approx 1.99939763739$

$a_6 \approx 1.99984940368$

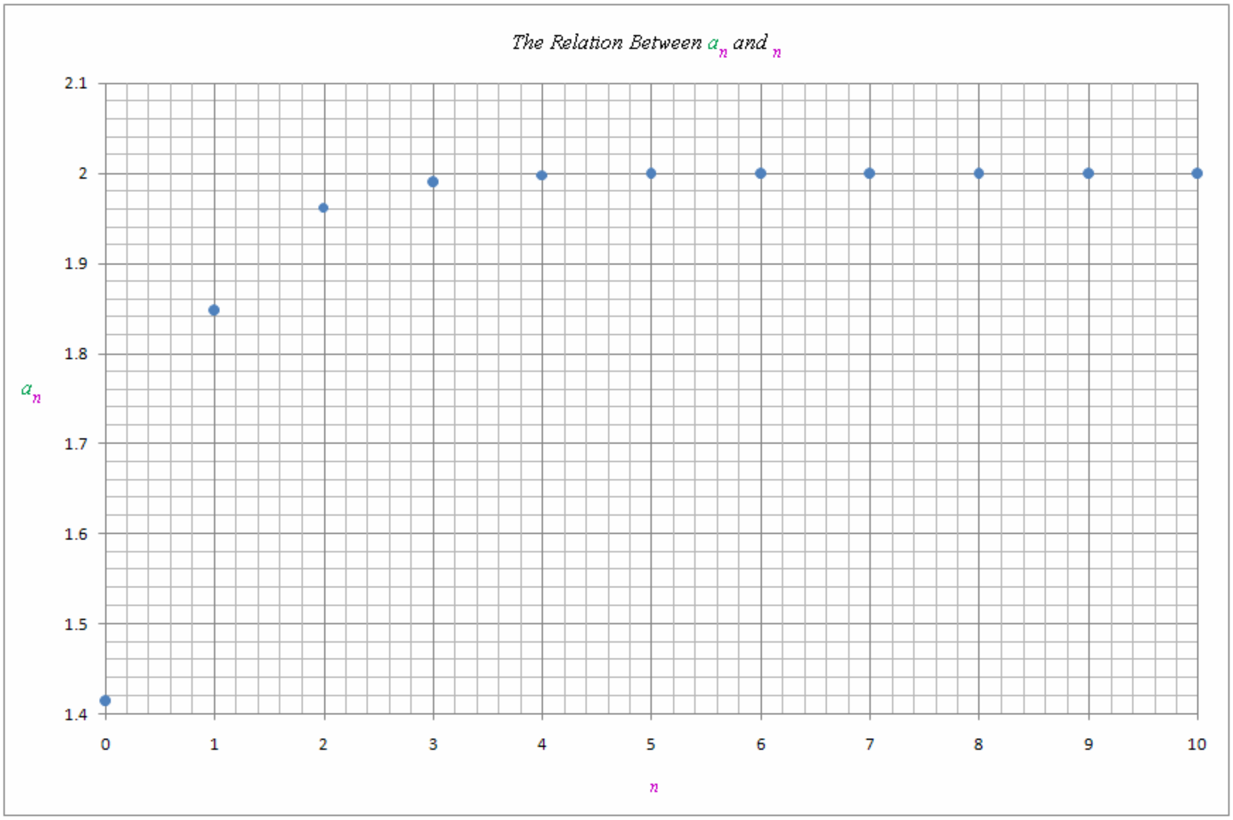
$a_7 \approx 1.99996235057$

$a_8 \approx 1.99999058762$

$a_9 \approx 1.99999764690$

$a_{10} \approx 1.99999941173$

Plotting the terms on a graph reveals a similar behavior as seen in the previous surd. The following graph illustrates the relation between n and a_n.



After the first five terms the data points increase by less and less suggesting an asymptote at 2. It is also clear that further points will not be much larger than a_{10} as the points are already on a horizontal trend. Also, as the values get larger the values of a_n and a_{n+1} will be nearly identical with a difference that is too miniscule to be significant.

This realization allows us to actually make them the same variable (x) in the previously deduced formula:

$$a_{n+1} = \sqrt{2 + a_n}$$

$$\boxed{x = \sqrt{2 + x}}$$

Giving us a new formula:

We can further simplify it:

$$(x = \sqrt{2 + x})^2$$

By squaring the equation, the negative solution is removed because a negative multiplied by a negative gives a positive.

$$x^2 = 2 + x$$

$$x^2 - x - 2 = 0$$

The variable can be isolated from the rest of the equation with the use of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where: a = 1, b= -1, c= -2

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{1 + \sqrt{9}}{2}$$

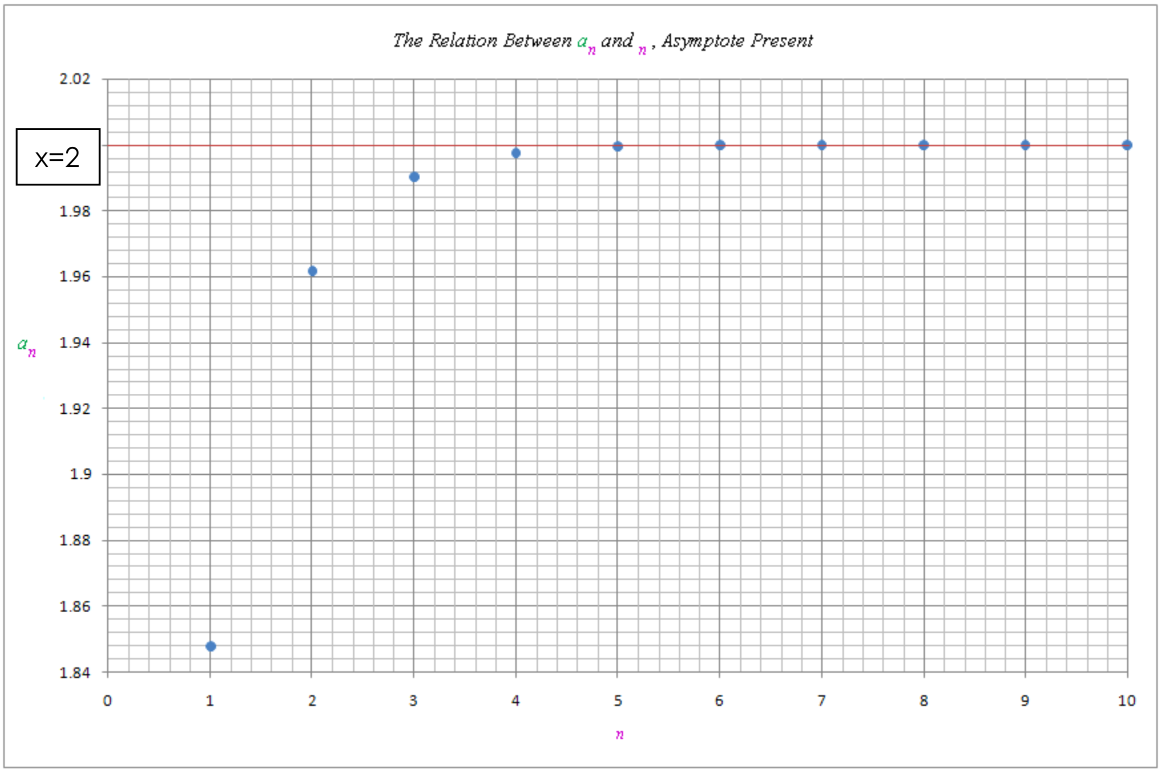
By taking squaring the equation we got rid of the negative solution therefore the solution to the infinite surd is:

$$\boxed{x=2}$$

The graph proves this as all the values are positive and the trend suggests that they will remain so.

This number appears to be the asymptote and is 2 just like predicted. The difference between the solution and a_{10} is only 0.00000058827 proving that further points are going to be almost the same as a_{10}.

By plotting the solution on the graph, we can see that the answer is where the asymptote would be.



Using previous methods a general formula will now be derived from the general surd:

$$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k}}}}\ldots$$

In terms of a_n the sequence of this surd can be expressed as:

$$a_1 = \sqrt{k + \sqrt{k}}$$

$$a_2 = \sqrt{k + \sqrt{k + \sqrt{k}}}$$

$$a_3 = \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}$$

$$a_4 = \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}$$

etc.

Since the only difference between one term and the next is an extra $(+\sqrt{k})$ a formula can be derived:

$$\boxed{a_{n+1} = \sqrt{k + a_n}}$$

Based on previous work, a_{n+1} and a_n can actually be considered the same as on the larger scale, the difference between larger values becomes unnoticeable. Through this, the previous formula can be rewritten as:

$$\boxed{x = \sqrt{k + x}}$$

We can further simplify it:

$(x = \sqrt{k + x})^2$

By squaring the equation, the negative solution is removed because a negative multiplied by a negative gives a positive.

$$x^2 = k + x$$

$$x^2 - x - k = 0$$

The variable can be isolated from the rest of the equation with the use of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where: a = 1, b= -1, c= -k

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-\mathrm{k})}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1 + 4\mathrm{k}}}{2}$$

Since once again the square root was squared, the negative answer can be removed leaving us with the general formula:

$$x = \frac{1 + \sqrt{1 + 4\mathrm{k}}}{2}$$

Using this formula, the answers of the first two surds are checked and appear to work. A table of values can be done to show the answers for the first twelve values of k.

K Value	Solution
1	1.618033988750
2	2
3	2.302775637732
4	2.561552812809
5	2.791287847478
6	3
7	3.192582403567
8	3.372281323269
9	3.541381265149

10	3.701562118716
11	3.854101966250
12	4

The values received are both integers and non-integers.

Integers are whole numbers including positives, negatives, and zero.

The first pattern that came to my mind is that between each solution that is an integer the number of non-integer solutions increases by two every time. As in, there are three non-integer solutions between 2 (k=2) and 3 (k=6). There are then five non-integer solutions between 3 (k=6) and 4 (k=12). To further test this I skipped seven values and tested k=20 which turns out to be 5, also an integer. This pattern will be further explored.

In order for the answer from the formula to be an integer, the numerator has to be an even number as it gets divided by 2 to give the solution. For this to happen, $\sqrt{1+4k}$ has to be odd because after 1 gets added to it, it has to be even so that it can be divisible by 2 while producing an integer. 4k +1 has to be a perfect square as well in order to yield a whole number when the root of it gets taken.

Since 4k+1 has to be a perfect square, the following formula can be derived where x is an integer:

$x^2 = 4k + 1$

From this we can calculate that: $k = \frac{x^2 - 1}{4}$

$x^2 - 1$ must be even then, so x^2 is therefore odd and as the result, so is x.

Since x is an odd integer, the formula for obtaining an odd integer can be used in to replace x:

$x = 2a + 1$

By putting this formula into the previously calculated expression ($k = \frac{x^2-1}{4}$), a new formula is formed:

$$k = \frac{(2a+1)^2 - 1}{4}$$

This formula can be simplified to give us:

$$k = \frac{4a^2 + 4a}{4}$$

$$k = a^2 + a$$

$$\boxed{k = a(a+1)}$$

Therefore k must be a product of two consecutive integers in order to provide an integer as the solution through the formula. Below is a table listing the first ten values of k that give an integer in the general formula used to find the value of a surd:

$$x = \frac{1 + \sqrt{1 + 4k}}{2}$$

K Valu	Solution
0 (0x1)	1
2 (1x2)	2
6 (2x3)	3
12 (3x4)	4
20 (4x5)	5
30 (5x6)	6
42 (6x7)	7
56 (7x8)	8
72 (8x9)	9
90 (9x10)	10

These values further support my claim. By substituting the formula for finding an odd integer with k in the general formula for finding the value of an infinite surd further proof is gained.

$$x = \frac{1 + \sqrt{1 + 4a(a+1)}}{2}$$

$$x = \frac{1 + \sqrt{4a^2 + 4a + 1}}{2}$$

$$x = \frac{1 + \sqrt{(2a+1)(2a+1)}}{2}$$

$$x = \frac{1 + (2a+1)}{2}$$

$$x = \frac{2a+2}{2}$$

$x = a + 1 \longrightarrow$ This can be any possible integer.

The value of k in the formula can be any integer, negatives and zero included. This is because the product of two consecutive integers will either give a positive number or zero.

A limitation on the claim is the **impossible production of 0 through the formula**. By working backwards, 0 can be put in place of x.

$$0 = \frac{1 + \sqrt{1 + 4k}}{2}$$

$$1 + \sqrt{1 + 4k} = 0$$

$$(\sqrt{1 + 4k})^2 = (-1)^2$$

$$1 + 4k = 1$$

$4k = 0$

$k = 0$

This solution is erroneous as the k value of 0 actually gives 1 through the formula. Sometimes zero is not considered an integer though so this is a possible limitation and exception to the formula.

The second major limitation is the **inability of the formula to produce negative integers**.

For example:

$$-2 = \frac{1 + \sqrt{1 + 4k}}{2}$$

$$-4 = 1 + \sqrt{1 + 4k}$$

$$(\sqrt{1 + 4k})^2 = (-5)^2$$

$$1 + 4k = 25$$

$$4k = 24$$

$$k = 6$$

The reason the formula does not work in producing negative integers is because the negative side of the equation at one point gets squared changing it to a positive.

The underlying reason for the inability of producing negatives is the fact that there cannot possibly be an infinite surd involving roots that produces negative integers.

root of a negative value gives an imaginary number which is part of the complex number system. This is because no two numbers of the same sign can be multiplied together to give a negative number.

The previously discussed pattern of increasing difference between k values that give integers in the answer can be easily justified.

For example: 0x1=0, 1x2=2, 2x3=6, 3x4=12, and 4x5=20.

It is clear that the difference in the products also increases by two as in the product of 0 and 1 is 0, and the product of 1 and 2 is 2. The difference between the two products is 2. The difference between the products of the next two consecutive integers then has a difference of 4.

The fact that this pattern exists, supports my claim that for an integer to be produced, k has to be the product of two consecutive integers in:

$$x = \frac{1 + \sqrt{1 + 4\mathrm{k}}}{2}$$

This formula appears to work with unlimited numbers as long as k is a product of two consecutive integers:

$$\frac{1 + \sqrt{1 + 4(10000000001 \times 10000000002)}}{2} = 10000000002$$