Math IA (SL TYPE I)

Circles

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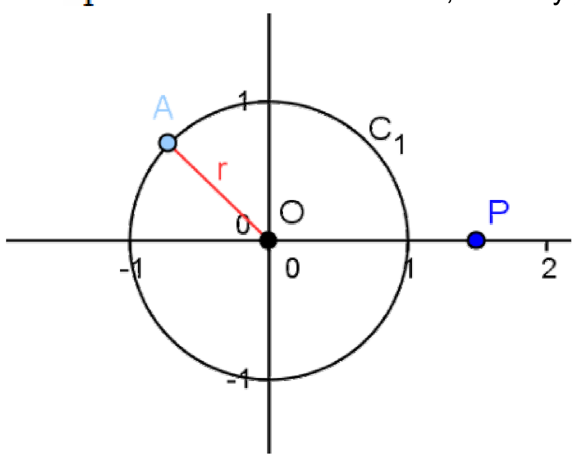
Aim: The aim of this task is to investigate positions of points in intersecting circles.

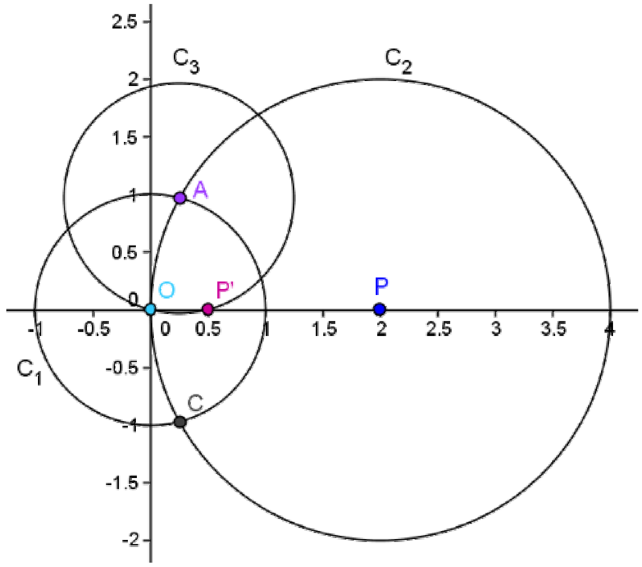
Introduction

The following diagram shows a circle C_1 with centre O and radius r, and any point P.



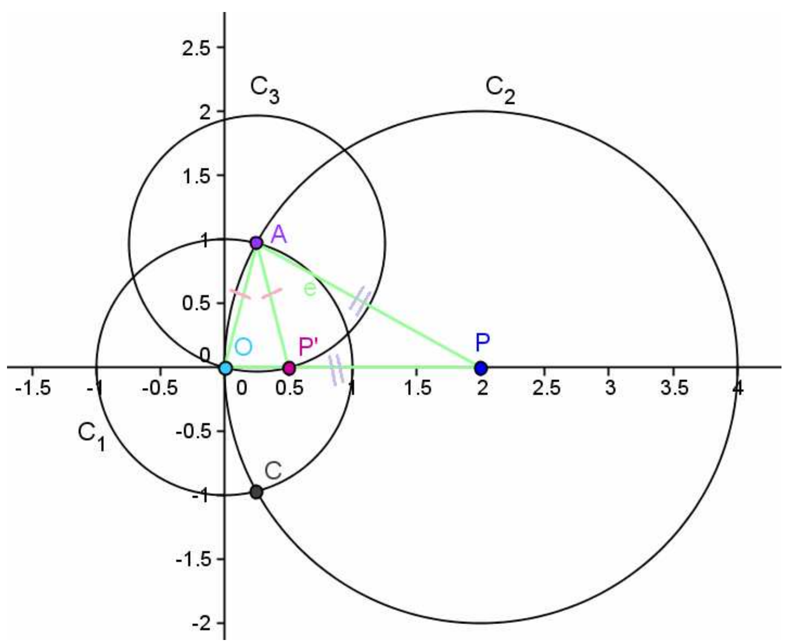
In this case, the r is the distance between any point (such as A) and the centre point O of the circle C_1. Since the radius is 1 unit, OP will also be 1 unit away from the point O.

The circle C_2 has centre P and radius $\overline{OP}$. A is one of the points of intersection of C_1 and C_2. Circle C_3 has centre A, and radius r. The point P' is the intersection of C_3 with (OP). The r=1, $\overline{OP}$=2, and P'=0.5. This is shown in the diagram below.



This investigation will explore in depth of the relationship between the r value and $\overline{\mathbf{OP}}$ values, when r is held constant and $\overline{\mathbf{OP}}$ values modified. It will also investigate the reverse, the relationship when the $\overline{\mathbf{OP}}$ values are held constant and the r values are modified.

In the first example, the r value given is 1. An analytic approach will be taken to find the length of $\overline{\mathbf{OP}}$’ when $\overline{\mathbf{OP}}$=2. Firstly, one can note that 2 isosceles triangle can be drawn by using the points A, O, P’, and P. It is shown in the diagram below.



In ∆AOP’, lines $\overline{\mathbf{OA}}$ and $\overline{\mathbf{AP'}}$ have the same length, because both points, O and P’ are within the circumference of the circle $\mathbf{C_3}$, which means that $\overline{\mathbf{OA}}$ and $\overline{\mathbf{AP'}}$ are its radius. Similarly, ∆AOP forms another isosceles triangle, because the lines $\overline{\mathbf{AP'}}$ and $\overline{\mathbf{OP}}$ are both radii of the circle $\mathbf{C_2}$.

$\overline{\mathbf{OA}}$= r of $\mathbf{C_3}$ or $\mathbf{C_1}$= 1. Since the circles are all graphed, they can be given coordinates. For $\overline{\mathbf{OA}}$, the coordinates of the point O will be (0, 0), because it lies in the origin of the graph. The coordinates of the point A is undefined, so we will substitute in variables for its values, (a, b). We have to find the coordinate of point A to find the coordinates of the point P'.

$\overline{\mathbf{AP}}$ = $\overline{\mathbf{OP}}$= r of $\mathbf{C_2}$= 2. We can get the coordinates for $\overline{\mathbf{AP}}$. The coordinates for point P will be (2, 0), for it lies on the x-axis and have a radius of 2. As for A, it is still undefined, so we will leave it as the unknown variables, (a, b).

By using the distance formula, $\sqrt{(\mathbf{a}-\mathbf{c})^2+(\mathbf{b}-\mathbf{d})^2}$, we can use a algebraic approach to system of equations to solve for the unknown coordinates. Through this, we can find out the coordinates of A.

$\overline{\mathbf{OA}}$= 1, A(a, b), O(0, 0)

Distance formula: $1= \sqrt{(a-0)^2+(b-0)^2}$

$1=\sqrt{a^2+b^2}$

$(1=\sqrt{a^2+b^2})^2$

$1= a^2+ b^2$

$\overline{\mathbf{AP}}$= 2, A(a, b), P(2, 0)

Distance formula: $2= \sqrt{(a-2)^2+(b-0)^2}$

$(2= \sqrt{(a^2+4a+4+b^2)})^2$

$4= a^2+4a+4+b^2$

Through the system of equations, the value of 'a' from the coordinates of point A can be found:

$$1= \cancel{a^2}+ \cancel{b^2}$$

$$- \quad \cancel{4}= \cancel{a^2}+4a+\cancel{4}+\cancel{b^2}$$

$$1= 4a$$

$$a= \frac{1}{4}$$

Then, through substitution, the value of 'b' can be found:

$$1= a^2+ b^2$$

$$1= \frac{1}{4}^2 + b^2$$

$$b^2 = 1 - \frac{1}{16}$$

$$= \frac{15}{16}$$

$$b = \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}$$

However, the negative value of b will be rejected, because in the graph, the value of b is in quadrant 1, where the y-value must be positive.

Now, the A(a, b) can be determined to A($\frac{1}{4}$,$\frac{\sqrt{15}}{4}$).

We must now find the coordinate of the point P' by using the distance formula. We will let P'= (c, 0). We know the length of $\overline{\mathbf{AP'}}$ because it is the radius of $\mathbf{C_3}$, which is 1. Since we know the coordinates of A, let's sub in the numbers in the distance formula.

$$\overline{\mathbf{AP'}} = 1 = \sqrt{\left(c - \frac{1}{4}\right)^2 + \left(0 - \frac{\sqrt{15}}{4}\right)^2}$$

$$= \sqrt{(c - \frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2}$$

$$[1 = \sqrt{\left(c - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\ ]^2$$

$$1 = (c - \frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2$$

$$1 - (\frac{\sqrt{15}}{4})^2 = (c - \frac{1}{4})^2$$

$$c - \frac{1}{4} = \sqrt{\frac{1}{16}}$$

$$c = \frac{1}{4} \pm \frac{1}{4}$$

$$c = \frac{1}{2}, 0$$

The 0 is rejected, because according to the graph shown before, the point P' does not lie on the origin. The coordinates of P' is ($\frac{1}{2}$, 0). Again, by using the distance formula, the length of $\overline{\mathbf{OP'}}$ can be calculated.

O(0, 0) and P'($\frac{1}{2}$, 0)

$$\sqrt{(0 - \frac{1}{2})^2 + (0-0)^2}$$

$$= \sqrt{(\frac{1}{2})^2} = \frac{1}{2}$$

$\therefore$ $\overline{\mathbf{OP}}$ = 2, r = 1, $\overline{\mathbf{OP}}' = \frac{1}{2}$

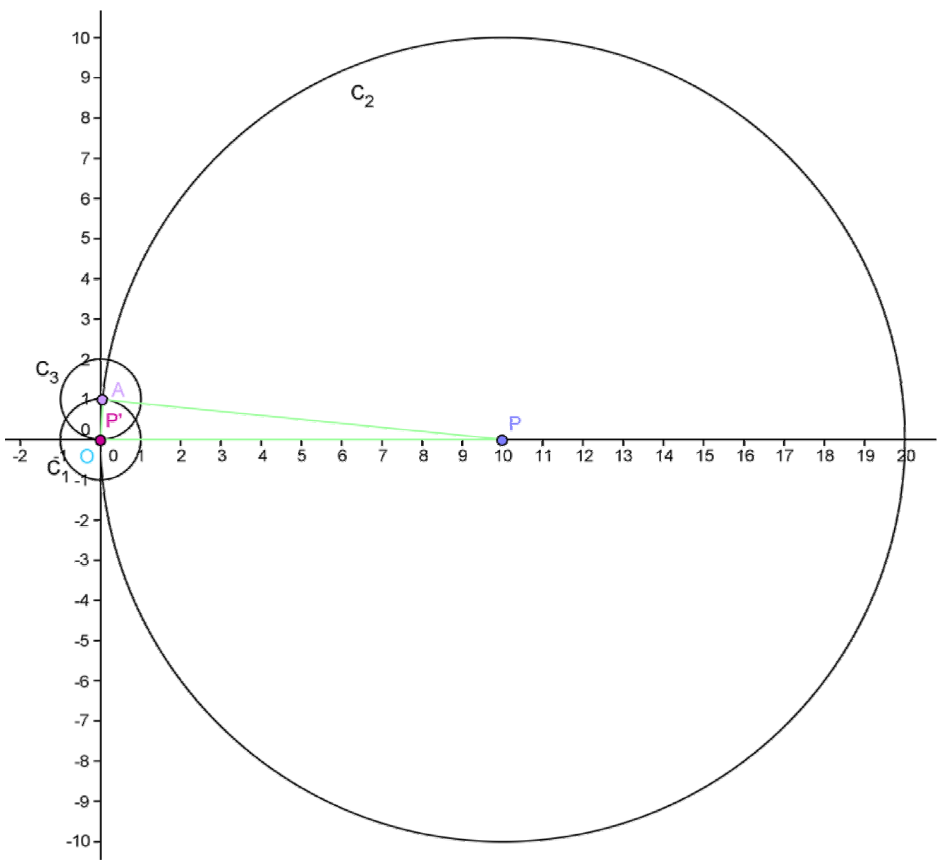
Using the same method of calculation, the following table was established.

r	$\overline{\mathbf{OP}}$	$\overline{\mathbf{OP}}'$
1	2	$\frac{1}{2}$
1	3	$\frac{1}{3}$
1	4	$\frac{1}{4}$

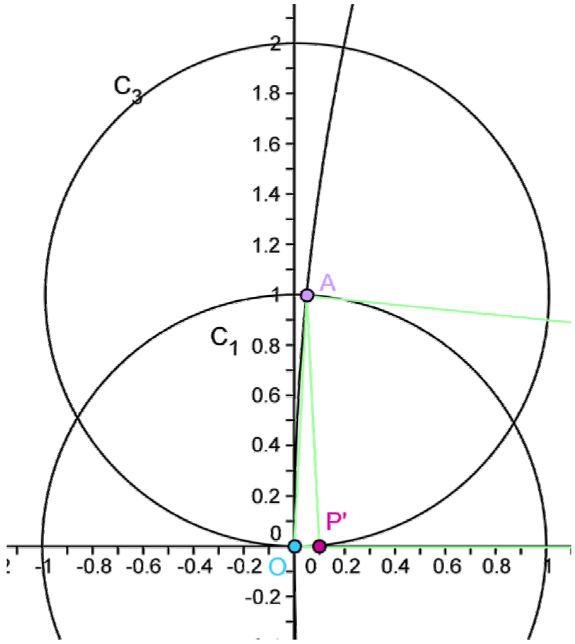
This pattern between the length of $\overline{\mathbf{OP}}$ and $\overline{\mathbf{OP}}'$ (when r is held constant) developed a relationship where they are inversely related ($\overline{\mathbf{OP}}' \propto \frac{1}{\overline{\mathbf{OP}}}$). Therefore, the general statement to represent this would be:

$$\overline{\mathbf{OP}}' = \frac{1}{\mathbf{n}}\text{ , (n= }\overline{\mathbf{OP}}\text{)}$$

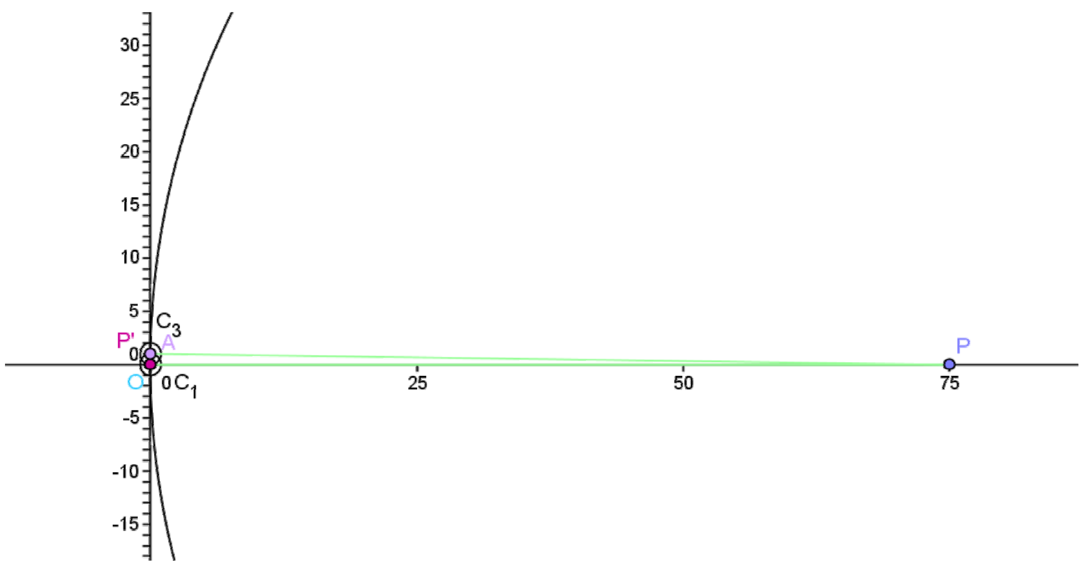
For checking the validity of the derived general statement, other values were checked. The general statement ($\overline{\mathbf{OP}}' = \frac{1}{\mathbf{n}}$, (n= $\overline{\mathbf{OP}}$).) still worked when r=1, $\overline{\mathbf{OP}}$= 10, $\overline{\mathbf{OP}}' = \frac{1}{10}$. In the following graph, r=1, $\overline{\mathbf{OP}}$= 10, $\overline{\mathbf{OP}}' = \frac{1}{10}$.



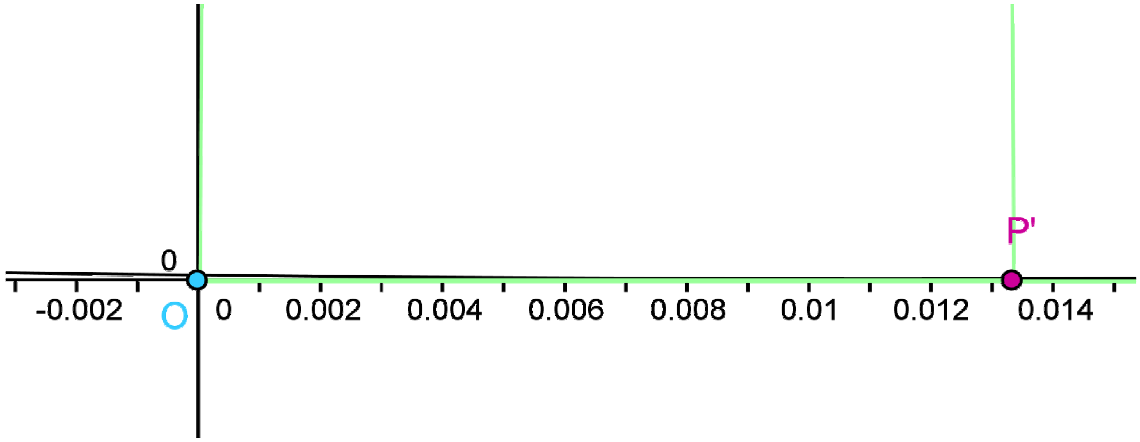
The following graph is a close up of the graph above, showing the value of $\overline{\mathbf{OP}}'$= $\frac{1}{10}$ = 0.1



A greater length of $\overline{\mathbf{OP}}$, such as 75 was calculated, and it still showed the result derived from the general statement, $\overline{\mathbf{OP}}' = \frac{1}{n}$, (n= $\overline{\mathbf{OP}}$). In the following graph, r=1, $\overline{\mathbf{OP}}$= 75, $\overline{\mathbf{OP}}'$= $\frac{1}{75}$



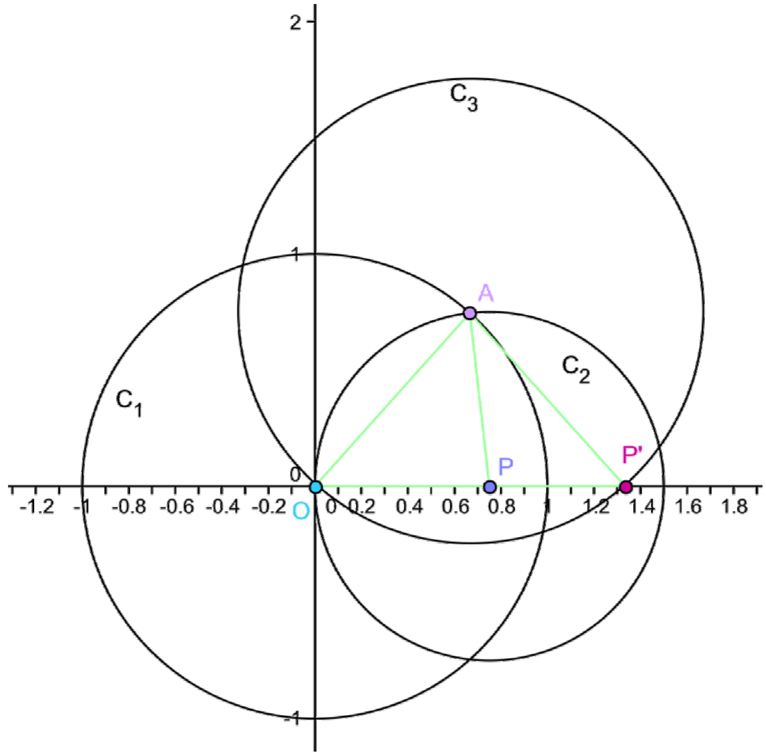
The following graph is a close-up of the graph above, showing the value of $\overline{\mathbf{OP}}'$= $\frac{1}{75}$ = $0.01\overline{3}$



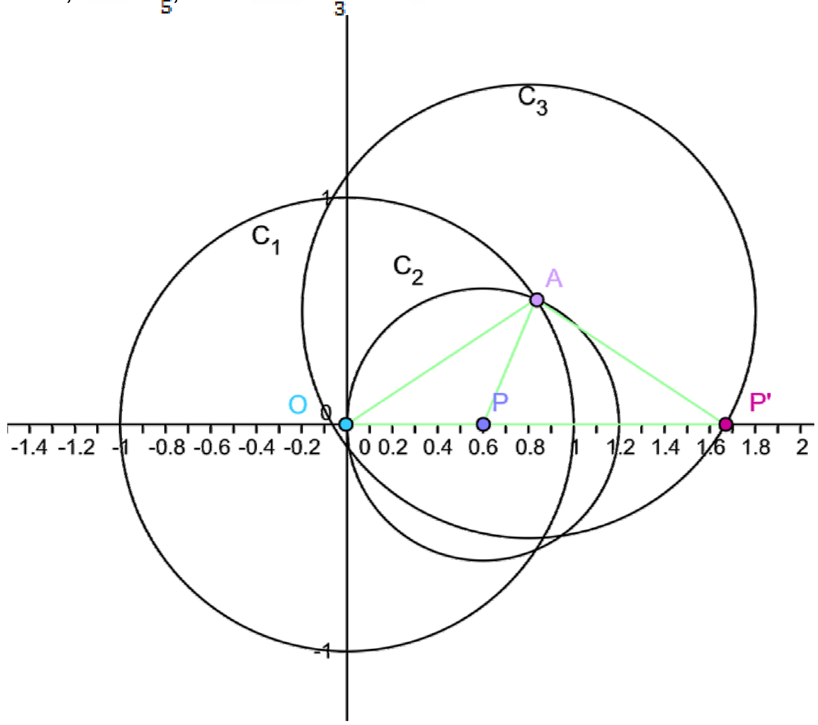
Thus, if 1 $\leq$ $\overline{\mathbf{OP}}$, the general statement will be valid.

To verify if the general statement is valid when the length of $\overline{\mathbf{OP}}$ $\leq$ 1, smaller values were also tried.

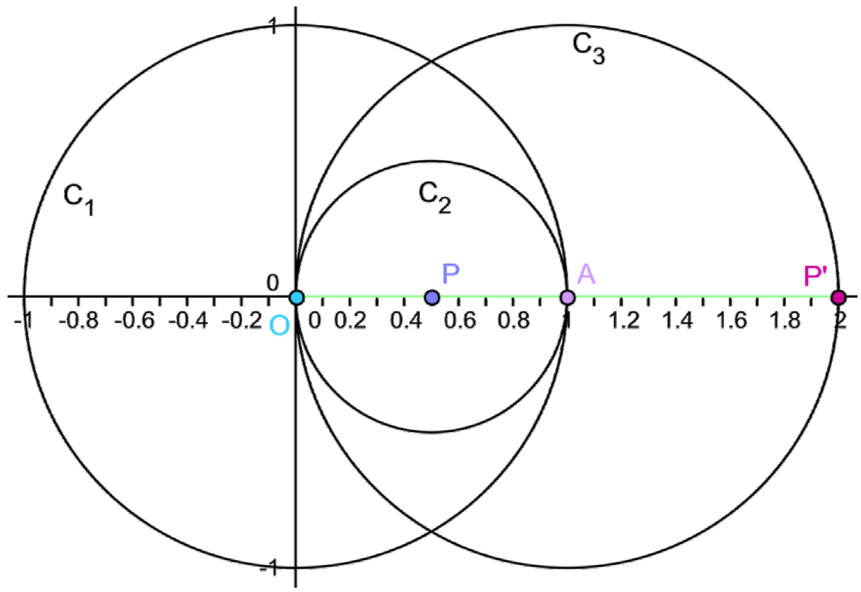
In this example, the length of $\overline{\mathbf{OP}}$= $\frac{3}{4}$, r= 1. Calculations resulted that $\overline{\mathbf{OP}}'$ still equaled to $\frac{4}{3}$. The following graph displayed is when r=1, $\overline{\mathbf{OP}}$= $\frac{3}{4}$, and $\overline{\mathbf{OP}}'$= $\frac{4}{3}$ = $1.\overline{3}$



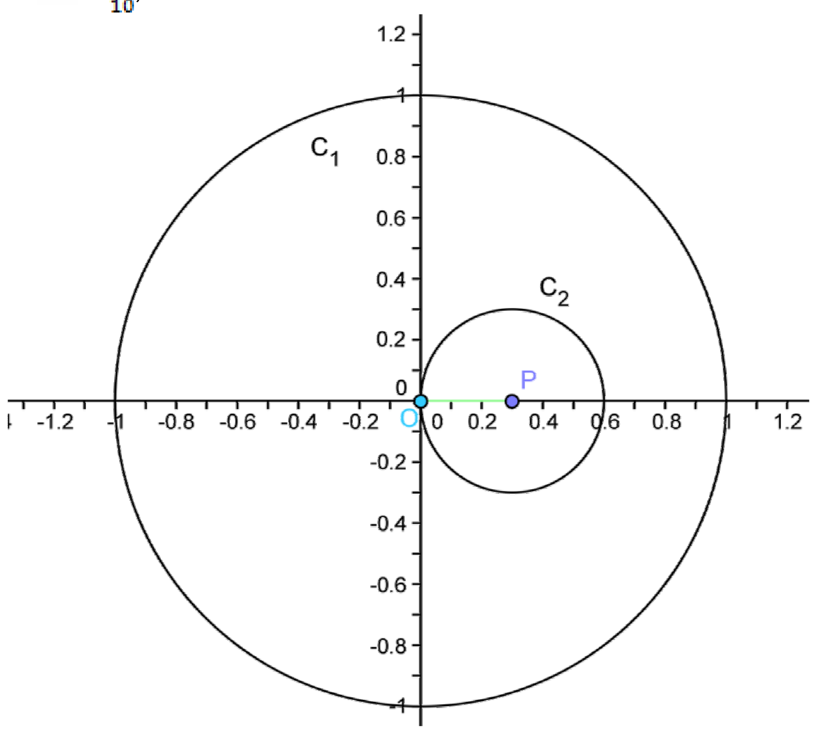
In the next example, the length of $\overline{\mathbf{OP}}$= $\frac{3}{5}$, r= 1. Calculations resulted that $\overline{\mathbf{OP}'}$ still equaled to $\frac{5}{3}$. The following graph displayed is when r= 1, $\overline{\mathbf{OP}}$= $\frac{3}{5}$, and $\overline{\mathbf{OP}}'$= $\frac{5}{3}$ = $1.\overline{6}$.



In the following graph, $\overline{\mathbf{OP}}$= $\frac{1}{2}$, r= 1. Through the same method done by the sample calculation, the $\overline{\mathbf{OP}}$'= 2, which makes the general statement still valid.



The following graph when $\overline{\mathbf{OP}}= \frac{3}{10}$, r= 1.

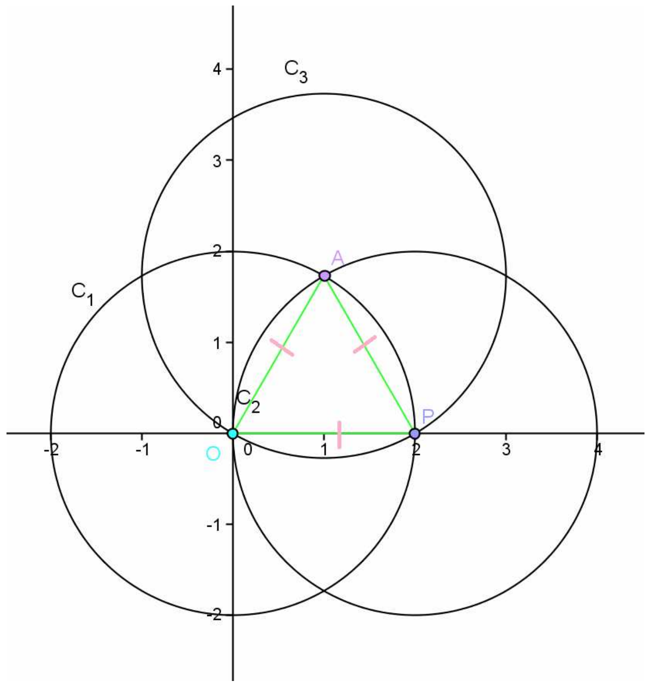


In the foreshown graph, circle $\mathbf{C_3}$ cannot be drawn, because there is no intersection point (point A) between $\mathbf{C_1}$ and $\mathbf{C_2}$. This makes sense, because if the radius of $\mathbf{C_2}$ is less than half the radius of $\mathbf{C_1}$, (in which this case is 0.5), then the circle won't large enough to intersect with $\mathbf{C_1}$.

Limitations in the general statement, $\overline{\mathbf{OP}}' = \frac{1}{\mathbf{n}}$, (n= $\overline{\mathbf{OP}}$) is that $\overline{\mathbf{OP}} \leq$ 0.5. The value 0.5 will only work for the case when r=1 and $\overline{\mathbf{OP}}$ values are changed. So in a broader sense, when radius is held constant, limitations to $\{\overline{\mathbf{OP}} | \ \overline{\mathbf{OP}} \leq \frac{\mathbf{r}}{2}, \quad \overline{\mathbf{OP}}$, r∈ N, natural number } . The $\overline{\mathbf{OP}}$(this is the radius of $\mathbf{C_2}$) and radius must be a natural number, because it cannot be 0, or be a negative number, because it refers to length.

The second part of the investigation will focus on finding a relationship of the length of $\overline{\mathbf{OP}'}$ when the length of $\overline{\mathbf{OP}}$ is held constant and the radius is changed.

The circle $\mathbf{C_2}$ has centre P and radius $\overline{\mathbf{OP}}$. A is one of the points of intersection of $\mathbf{C_1}$ and $\mathbf{C_2}$. Circle $\mathbf{C_3}$ has centre A, and radius r. The point P' is the intersection of $\mathbf{C_3}$ with (OP). The r=2, $\overline{\mathbf{OP}}$=2. In this case, the P' and P have the same coordinates. It is shown in the following graph.



You can notice that an equilateral triangle can be formed. ∆AOP is an equilateral triangle, because the radius of all three circles, C_1, C_2, and C_3 are all 2. In this case, to find the length of $\overline{\mathbf{OP}}$', it is the same length as $\overline{\mathbf{OP}}$, because they lie on the same coordinate. So, the length would be 2. However, through an analytical approach, and the use of the distance formula and the systems of equation, it can be proved.

We know that $\overline{\mathbf{OA}}$ = the radius of $\mathbf{C_3}$. The point O lies on the origin, so its coordinates would be O(0, 0). However, point A is unknown, so variables, (a, b), will be substituted for its values. $\overline{\mathbf{AP}}$ = radius of $\mathbf{C_2}$. Since point P lies on the origin, its coordinates are P(2, 0). Again, point A is unknown, A(a, b).
By using the distance formula, $\sqrt{(\mathbf{a}-\mathbf{c})^2+(\mathbf{b}-\mathbf{d})^2}$, we can use a algebraic approach to system of equations to solve for the unknown coordinates. Through this, we can find out the coordinates of A.
$\overline{\mathbf{OA}}$ = 2, O(0, 0), A(a, b)

Distance formula: 2= $\sqrt{(a-0)^2+(b-0)^2}$

$$2 = \sqrt{a^2 + b^2}$$

$$(2 = \sqrt{a^2 + b^2})^2$$

$$4 = a^2 + b^2$$

$\overline{\mathbf{AP}}$ = 2, P(2, 0), A(a, b)

$$\text{Distance formula: } 2 = \sqrt{(a-2)^2 + (b-0)^2}$$

$$2 = \sqrt{a^2 - 4a + 4 + b^2}$$

$$(2 = \sqrt{a^2 - 4a + 4 + b^2})^2$$

$$4 = a^2 - 4a + 4 + b^2$$

Through the system of equations, the value of 'a' from the coordinates of point A can be found:

$$4 = \cancel{a^2} + \cancel{b^2}$$

$$- \quad \cancel{4} = \cancel{a^2} - 4a + \cancel{4} + \cancel{b^2}$$

$$4 = 4a$$

$$a = 1$$

Then, through substitution, the value of 'b' can be found:

$$4 = a^2 + b^2$$

$$4 = 1^2 + b^2$$

$$b^2 = 4 - 1$$

$$b = \pm\sqrt{3}$$

However, the negative value of b will be rejected, because in the graph, the value of b is in quadrant 1, where the y-value must be positive.

Now, the A(a, b) can be determined to A($1, \sqrt{3}$).

We must now find the coordinate of the point P' by using the distance formula. We will let P'= (c, 0). We know the length of $\overline{\mathbf{AP'}}$ because it is the radius of $\mathbf{C_3}$, which is 2. Since we know the coordinates of A, let's sub in the numbers in the distance formula.

$\overline{\mathbf{AP'}}$ = 2, A($1, \sqrt{3}$), P'(c, 0)

$$\text{Distance formula: } 2 = \sqrt{(c-1)^2 + (\sqrt{3} - 0)^2}$$

$$(2 = \sqrt{(c-1)^2 + (\sqrt{3} - 0)^2})^2$$

$$4 = (c-1)^2 + 3$$

$$0 = -1 + c^2 - 2c + 1$$

$$0 = c^2 - 2c$$

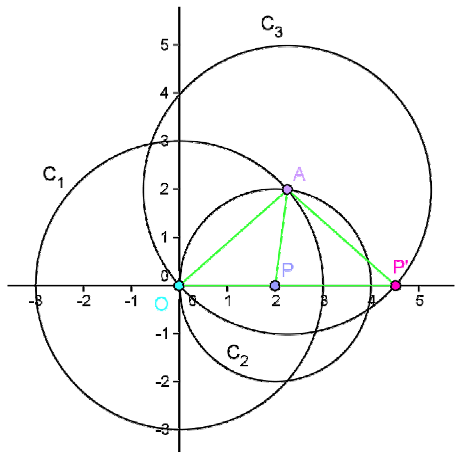
$$0 = c(c-2)$$

c= 2, 0

There are potentially two values for c. However, 0 is rejected, because in the graph shown above, P'=P, the point does not lie at the origin. $\therefore$ P'(2, 0). Again, by using the distance formula, the length of $\overline{\mathbf{OP}}'$ can be calculated. O(0, 0), P'(2,0)

$$\overline{\mathbf{OP}}' = \sqrt{(2-0)^2+(0-0)^2}$$
$$= \sqrt{2^2} = 2$$
$$\therefore\ \overline{\mathbf{OP}} = 2,\ r = 2,\ \overline{\mathbf{OP}}' = 2$$

The following graph displayed is when $\overline{\mathbf{OP}} =$ 2, r= 2. Through the same method of calculation as the above, $\overline{\mathbf{OP}}' = \frac{9}{2}$.



Using the same method of calculation, the following chart was established

r	$\overline{\mathbf{OP}}$	$\overline{\mathbf{OP}}$'
1	2	$\frac{1}{2}$
2	2	$\frac{4}{2}$= 2
3	2	$\frac{9}{2}$
4	2	$\frac{16}{2}$= 8

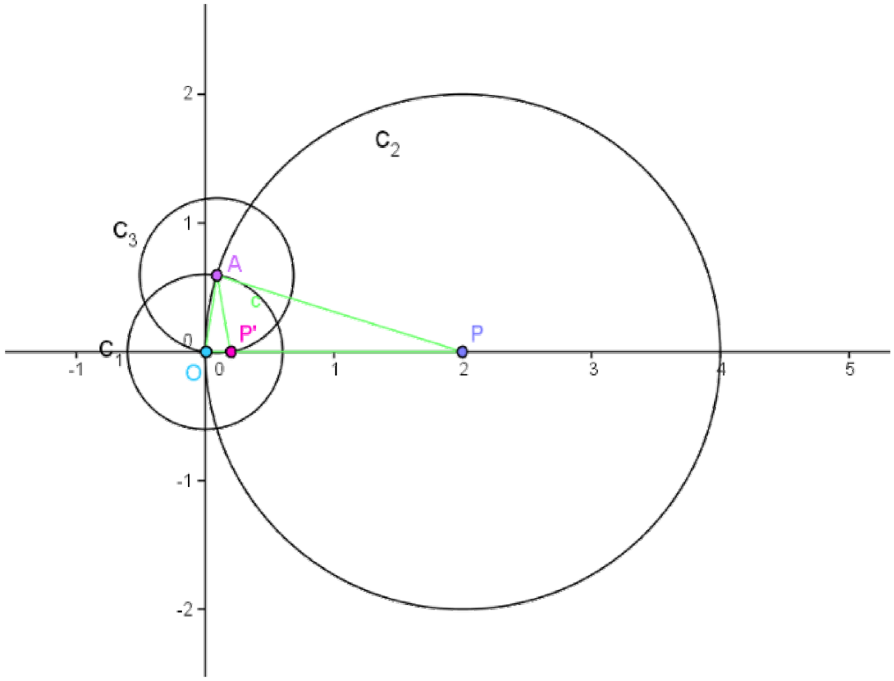
This pattern between the length of $\overline{\mathbf{OP}}$ and $\overline{\mathbf{OP}}'$ (when $\overline{\mathbf{OP}}$ is held constant) develops an exponential relationship ($\overline{\mathbf{OP}}$' $\propto$ $\frac{(\overline{\mathbf{OP}})^2}{2}$). Therefore, the general statement to represent this would be:

$$\overline{\mathbf{OP}}\text{'} = \frac{n^2}{2}\ \text{(n= r)}$$

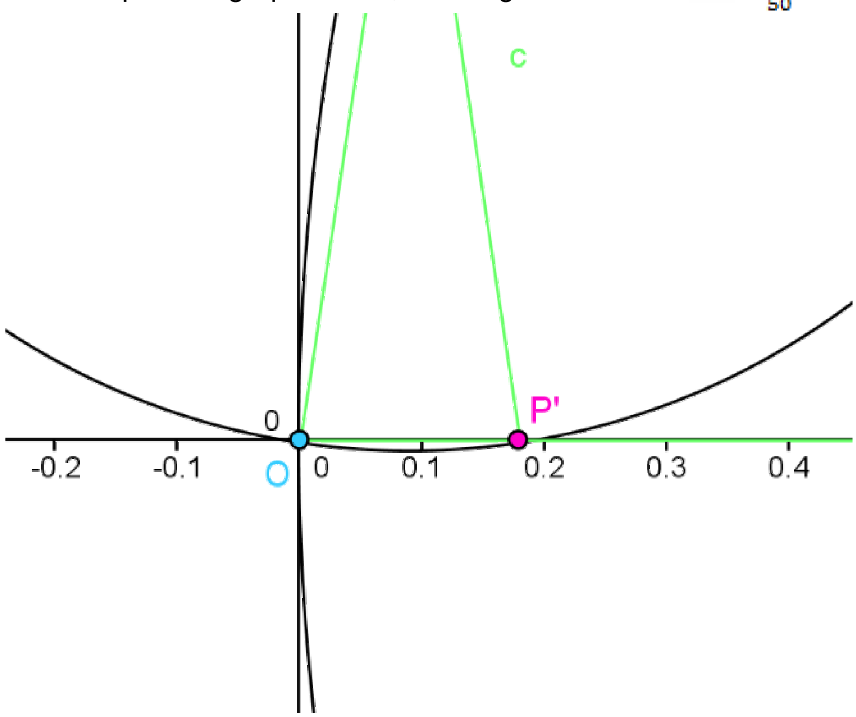
To check the validity of the general statement, other values for r were also calculated.

In the following graph, $\overline{\mathbf{OP}} =$ 2, r = $\frac{3}{5}$. Through the same method of calculation as the sample calculation,

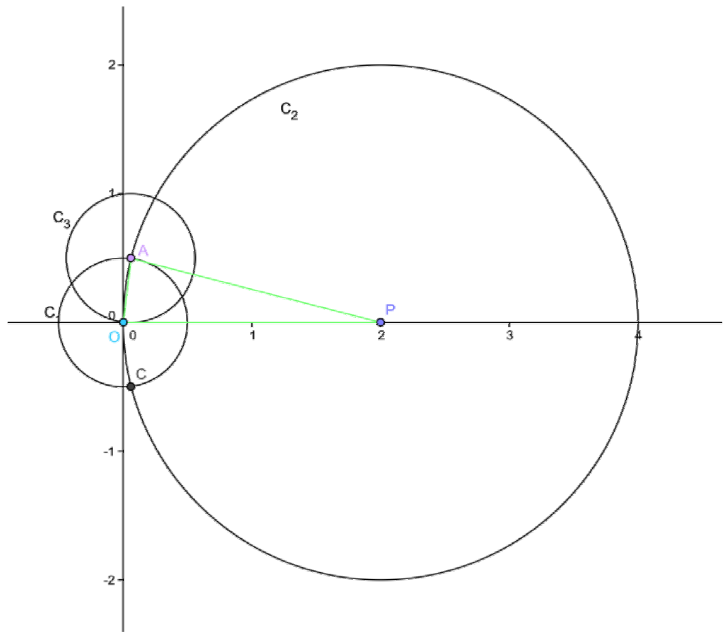
$\overline{\mathbf{OP}}' = \frac{9}{50}$, which validates the general statement.



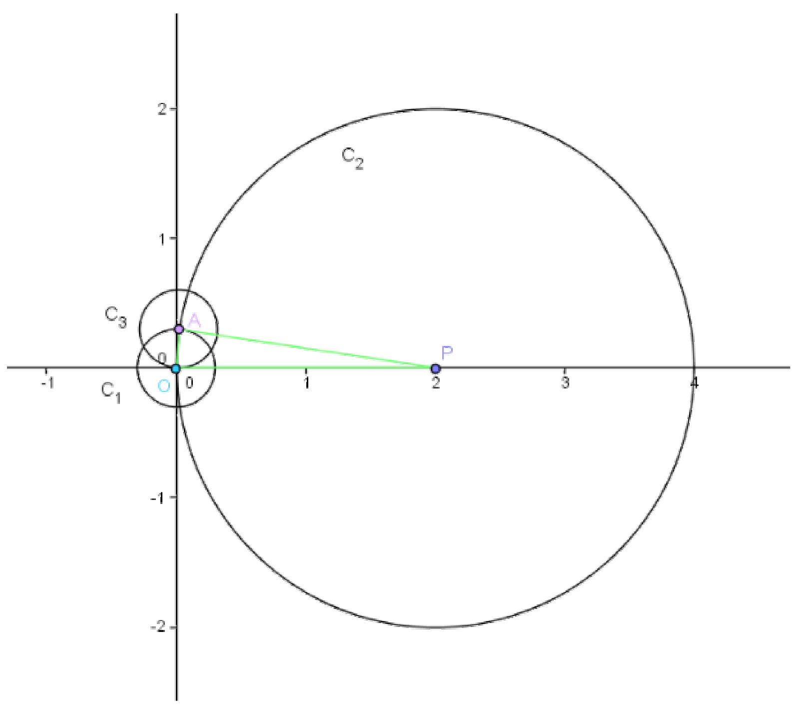
The following graph is a close-up of the graph above, showing the value of $\overline{\mathbf{OP}}' = \frac{9}{50} = 0.18$.



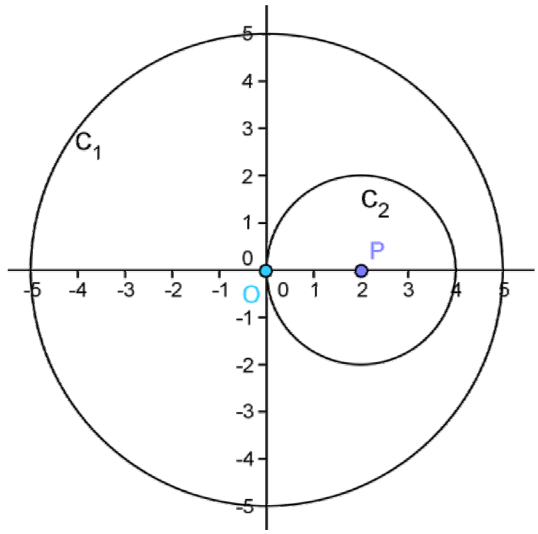
In the following graph, $\overline{\mathbf{OP}} = 2$, r = $\frac{1}{2}$. $\mathbf{C_3}$ can be drawn from the centre point A($\frac{3}{50}$, $\frac{1}{2}$). However, there is no P', for $\mathbf{C_3}$ and $\overline{\mathbf{OP}}$ do not intersect.



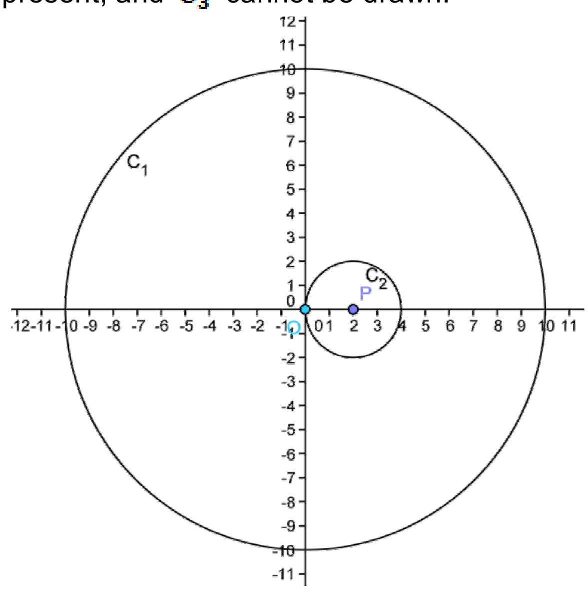
Another smaller value of r, in which $\overline{\mathbf{OP}} =$ 2, r = $\frac{3}{10}$ is shown in the following graph. Similar to the foreshown graph, $\mathbf{C_3}$ can be drawn from the centre point A($\frac{1}{50}$, $\frac{3}{10}$). However, there is no P', for $\mathbf{C_3}$ and $\overline{\mathbf{OP}}$ do not intersect.



In the following graph, $\overline{\mathbf{OP}} =$ 2, r =5. $\mathbf{C_1}$ and $\mathbf{C_2}$ does not intersect, so there is no Point A. $\mathbf{C_3}$ cannot be created without A as its centre point. Therefore, there is no point P', and the length of $\overline{\mathbf{OP}}$' cannot be determined.



Similarly, like the graph above, the following graph is when $\overline{\mathbf{OP}} =$ 2, r =10. Again, because the circles, $\mathbf{C_1}$ and $\mathbf{C_2}$ don't intersect, point A is not present, and $\mathbf{C_3}$ cannot be drawn.



It is interesting to note that the general statement is still valid up to r= 1, 2, 3, 4 but after when r > 4, there was no point of intersection between $\mathbf{C_1}$ and $\mathbf{C_2}$. In addition, when r $\le \frac{1}{2}$, there was a point of intersection (A) between $\mathbf{C_1}$ and $\mathbf{C_2}$, so $\mathbf{C_3}$ could be drawn, but there was no point of intersection between $\overline{\mathbf{OP}}$ and $\mathbf{C_3}$. Thus, the limitation for the general formula $\overline{\mathbf{OP}}$' = $\frac{\mathbf{n^2}}{\mathbf{2}}$ (n= r) would be 4 $\le$ r < 0.5.

These values only work if $\overline{\mathbf{OP}}$ = 2, so the general limitation would be $\{$ r $|(\overline{\mathbf{OP}})^{-1} <$ r $\le (\overline{\mathbf{OP}})^2\ \overline{\mathbf{OP}}, \mathbf{r} \in$ N, Natural Numbers$\}$ The radius must be a natural number, because the length of something cannot be 0 or have a negative value, for it is a measurement.

The two general statements derived from this investigation are:

$$\overline{\mathbf{OP}}' = \frac{\mathbf{1}}{\mathbf{n}} \text{, (n= } \overline{\mathbf{OP}}\text{)}$$

and

$$\overline{\mathbf{OP}}' = \frac{n^2}{2} \text{ (n= r).}$$

The first general statement is consistent with the second general statement. It is why when you calculate the lengths of $\overline{\mathbf{OP}}'$, you can use the same method for both situations. However, the derived general statement appears different, because each one is dealing with a different constant. The first general statement is when the radius of $\mathbf{C_1}$ and $\mathbf{C_2}$ is held constant, with changing radius of $\mathbf{C_3}$ ($\overline{\mathbf{OP}}$), while the second general statement is when the radius of $\mathbf{C_1}$ and $\mathbf{C_2}$ are modified, while the radius of $\mathbf{C_3}$ ($\overline{\mathbf{OP}}$) remains constant.

Conclusion

r	$\overline{\mathbf{OP}}$	$\overline{\mathbf{OP}}'$
1	$\frac{3}{10}$	Not defined
1	$\frac{1}{2}$	2
1	$\frac{3}{5}$	$\frac{5}{3}$
1	$\frac{3}{4}$	$\frac{4}{3}$
1	$\frac{4}{5}$	$\frac{5}{4}$
1	1	1
1	2	$\frac{1}{2}$
1	3	$\frac{1}{3}$
1	4	$\frac{1}{4}$
1	5	$\frac{1}{5}$
1	100	$\frac{1}{100}$
1	1000	$\frac{1}{1000}$

$\overline{\mathbf{OP}}$ and $\overline{\mathbf{OP}}'$ (when r is held constant) are inversely related ($\overline{\mathbf{OP}}' \propto \frac{1}{\overline{\mathbf{OP}}}$). Therefore, the general statement to represent this would be:

$$\overline{\mathbf{OP}}' = \frac{1}{n} \text{ , (n= } \overline{\mathbf{OP}}\text{)}$$

Limitations in this general statement, when radius is held constant, is $\{\overline{\mathbf{OP}} \mid \overline{\mathbf{OP}} \le \frac{r}{2}$, r$\in$ N, natural number$\}$. If the radius of $\mathbf{C_2}$ is less than half the radius of $\mathbf{C_1}$, then the circle won't be large enough to intersect with $\mathbf{C_1}$. The radius must be a natural number, because it cannot be 0, or be a negative number, because it refers to length.

r	$\overline{OP}$	$\overline{OP}$'
$\frac{3}{10}$	2	Not defined
$\frac{1}{2}$	2	Not defined
$\frac{2}{5}$	2	$\frac{9}{50}$
1	2	$\frac{1}{2}$
2	2	$\frac{4}{2}$= 2
3	2	$\frac{9}{2}$
4	2	$\frac{16}{2}$= 8
5	2	Not defined

$\overline{OP}$ and $\overline{OP}'$ (when $\overline{OP}$ is held constant) is an exponential relationship ($\overline{OP}' \propto \frac{(\overline{OP})^2}{2}$). Therefore, the general statement to represent this would be:

$$\overline{OP}' = \frac{n^2}{2} \text{ (n= r).}$$

Limitations in this general statement would be $\{r \mid (\overline{OP})^{-1} < r \leq (\overline{OP})^2\ \overline{OP}, r \in$ N, Natural Numbers$\}$. If the radius is greater than length of $\overline{OP}$'s squared value, then there wouldn't be any point of intersection between C_1 and C_2. When the radius is less than the inverse of $\overline{OP}$, C_3 and $\overline{OP}$ do not intersect.