Mathematics Standard Level
Portfolio Assignment Type II
Logarithm bases

Lukasz Weclas

December 2009

Table of Contents

Introduction

Logarithm is defined as the exponent that indicates the power to which a base number is raised to produce a given number[1]. In this assignment I shall attempt to investigate the characteristics of sequences of logarithms. As a conclusion, I will try to find the general statement and finally the range and limitations of *a, b* and *x* will be considered.

Write down the next two terms of each sequence

The given examples of logarithms are as follows:

$A: \log_2 8, \log_4 8, \log_8 8, \log_{16} 8, \log_{32} 8...$

$B: \log_3 81, \log_9 81, \log_{27} 81, \log_{81} 81, ...$

$C: \log_5 25, \log_{25} 25, \log_{125} 25, \log_{625} 25, ...$

:

:

$X: \log_m m^k, \log_{m^2} m^k, \log_{m^3} m^k, \log_{m^4} m^k, ...$

It can be easily noticed that in each sequence every term has the same number of logarithms, respectively:

A: 8

B: 81

C: 25

X: m^k

---

[1] http://m-w.com/dictionary/logarithm

What is changing in the sequences of logarithms are the bases of logarithms. However the similarity presents, as they can be all written down as $a_n = \log_{x^n} Y$. In the given data the value of x is respectively:

A: 2

B: 3

C: 5

X: m

Knowing this two thing, I can now calculate and write down the next two terms of each sequence:

A: $a_6 = \log_{2^6} 8 = \log_{64} 8$, $a_7 = \log_{2^7} 8 = \log_{128} 8$

B: $a_5 = \log_{3^5} 81 = \log_{243} 81$, $a_6 = \log_{3^6} 81 = \log_{729} 81$

C: $a_5 = \log_{5^5} 25 = \log_{3125} 25$, $a_6 = \log_{5^6} 25 = \log_{15625} 25$

X: $a_5 = \log_{m^5} m^k$, $a_6 = \log_{m^6} m^k$

Find an expression for the n^{th} term of each sequence and write in the form $\frac{p}{q}$

Using the knowledge gained in previous task an expression for the n^{th} term of each sequence can be calculated:

A: $a_n = \log_{2^n} 8$

B: $a_n = \log_{3^n} 81$

C: $a_n = \log_{5^n} 25$

X: $a_n = \log_{m^n} m^k$

One of the possible ways to present these expressions in the form $\frac{p}{q}$,where p, q $\in$Z, is to use formulas $\log_a b = \frac{\log_c b}{\log_c a}$ (for base substitution) and $\log_a b^x = x\log_a b$. The answers will be justified with the use of GDC Casio CFX-9850GB PLUS.

A: $a_n = \log_{2^n} 8 = \log_{2^n} 2^3 = \frac{\log 2^3}{\log 2^n} = \frac{3\log 2}{n\log 2} = \frac{3}{n}$

Now I substitute n for any given number, for example 15. Let n=15

$$a_{15} = \log_{2^{15}} 8 = \log_{2^{15}} 2^3 = \frac{\log 2^3}{\log 2^{15}} = \frac{3\log 2}{15\log 2} = \frac{3}{15} = \frac{1}{5} = 0.2$$

To justify my answer, I will use my GDC to check if it is correct

$a_{15} = \log_{32768} 8 = \frac{\log 8}{\log 32768} = 0.2$

The answer is the same, therefore it is correct.

B: $a_n = \log_{3^n} 81 = \log_{3^n} 3^4 = \frac{\log 3^4}{\log 3^n} = \frac{4\log 3}{n\log 3} = \frac{4}{n}$

Now I substitute n for any given number, for example 12. Let n=16

$a_{16} = \log_{3^{16}} 81 = \log_{3^{16}} 3^4 = \frac{\log 3^4}{\log 3^{16}} = \frac{4\log 3}{16\log 3} = \frac{4}{16} = 0.25$

To justify my answer, I will use my GDC to check if it is correct

$a_{16} = \log_{43046721} 81 = \frac{\log 81}{\log 43046721} = 0.25$

The answer is the same, therefore it is correct.

$$\text{C: } a_n = \log_{5^n} 25 = \log_{5^n} 5^2 = \frac{\log 5^2}{\log 5^n} = \frac{2\log 5}{n\log 5} = \frac{2}{n}$$

Now I substitute n for any given number, for example 8. Let n=8

$$a_8 = \log_{5^8} 25 = \log_{5^8} 5^2 = \frac{\log 5^2}{\log 5^8} = \frac{2\log 5}{8\log 5} = \frac{2}{8} = 0.25$$

To justify my answer, I will use my GDC to check if it is correct

$$a_8 = \log_{390625} 25 = \frac{\log 25}{\log 390625} = 0.25$$

The answer is the same, therefore it is correct.

$$\text{X: } a_n = \log_{m_n} m^k = \frac{\log m^k}{\log m^n} = \frac{k\log m}{n\log m} = \frac{k}{n}$$

Conducting the same process is not possible in case of sequence X, but the three equations above should prove that calculation would be possible if k, n $\in$ □ .

<u>Calculate the value of given logarithms</u>

The next task is to calculate the following sequences given in the assignment, then give my answers in the form $\frac{p}{q}$, where p, q $\in$ □ :

$$D: \log_4 64, \log_8 64, \log_{32} 64$$

$$E: \log_7 49, \log_{49} 49, \log_{343} 49$$

$$F: \log_{\frac{1}{5}} 125, \log_{\frac{1}{125}} 125, \log_{\frac{1}{625}} 125$$

$$G: \log_8 512, \log_2 512, \log_{16} 512$$

The given sequences are similar to the ones given in previous task. I will use the first example whether the formula $a_n = \log_{m^n} a^n = \log_{m^n} m^k = \frac{k}{n}$ (conclusion from previous task) can be used here.

D: $a_2 = \log_4 64 = \log_{m^n} m^k = \log_{2^2} 2^6 = \frac{\log 2^6}{\log 2^2} = \frac{6\log 2}{2\log 2} = \frac{6}{2} = 3$

The formula can be used in this calculations, therefore I can proceed.

$$a_3 = \log_8 64 = \log_{2^3} 2^6 = \frac{\log 2^6}{\log 2^3} = \frac{6\log 2}{3\log 2} = \frac{6}{3} = 2$$

$$a_5 = \log_{32} 64 = \log_{2^5} 2^6 = \frac{\log 2^6}{\log 2^5} = \frac{6\log 2}{5\log 2} = \frac{6}{5} = 1.2$$

E: $a_1 = \log_7 49 = \log_{7^1} 7^2 = \frac{\log 7^2}{\log 7^1} = \frac{2\log 7}{1\log 7} = \frac{2}{1} = 2$

$$a_2 = \log_{49} 49 = \log_{7^2} 7^2 = \frac{\log 7^2}{\log 7^2} = \frac{2\log 7}{2\log 7} = \frac{2}{2} = 1$$

$$a_3 = \log_{343} 49 = \log_{7^3} 7^2 = \frac{\log 7^2}{\log 7^3} = \frac{2\log 7}{3\log 7} = \frac{2}{3}$$

F:

$$a_1 = \log_{\frac{1}{5}} 125 = \log_{5^{-1}} 5^3 = \frac{\log 5^3}{\log 5^{-1}} = \frac{3\log 5}{-1\log 5} = \frac{3}{-1} = -3$$

$$a_3 = \log_{\frac{1}{125}} 125 = \log_{5^{-3}} 5^3 = \frac{\log 5^3}{\log 5^{-3}} = \frac{3\log 5}{-3\log 5} = \frac{3}{-3} = -1$$

$$a_4 = \log_{\frac{1}{625}} 125 = \log_{5^{-4}} 5^3 = \frac{\log 5^3}{\log 5^{-4}} = \frac{3\log 5}{-4\log 5} = \frac{3}{-4} = -0.75$$

$$G: a_3 = \log_8 512 = \log_{2^3} 2^9 = \frac{\log 2^9}{\log 2^3} = \frac{9\log 2}{3\log 2} = \frac{9}{3} = \frac{3}{1} = 3$$

$$a_1 = \log_2 512 = \log_{2^1} 2^9 = \frac{\log 2^9}{\log 2^1} = \frac{9\log 2}{1\log 2} = \frac{9}{1} = 9$$

$$a_4 = \log_{16} 512 = \log_{2^4} 2^9 = \frac{\log 2^9}{\log 2^4} = \frac{9\log 2}{4\log 2} = \frac{9}{4} = 2.25$$

Describe how to obtain the third answer in each row from the first two answers

From the calculations above two conclusions can be drawn necessary to obtain the third answer in each row from the first two answers. Firstly, the argument of logarithm is the same in each sequence (64, 49, 125 and 512 for D, E, F for G respectively). Secondly, it can be noticed that third base of logarithm is a product of the first two (example E: 7 × 49 = 343). Therefore a formula can be created for a_1 being the base of first logarithm, a_2 being the base of second logarithm and Q being argument of logarithm, such as $a_3 = \log_{a_1 a_2} K$. As a confirmation, I shall use example E:

$$a_3 = \log_{343} 49 = \log_{7\times 49} 49 = \frac{2}{3}$$

But in the given task, there is one more possibility of obtaining the third answer in each row from the first two answers. It can be noticed that numerator of the third answer is a product of first and second answers, when its denominator is a sum of first and second answer. I shall use the same example to present it:

$$a_3 = \log_{343} 49 = \frac{2}{3} = \frac{2\times 1}{2+1} = \frac{\log_7 49 \times \log_{49} 49}{\log_7 49 + \log_{49} 49}$$

Create two more examples that fit the pattern

To study the pattern deeply it is vital to create two more examples that will show whether or not the formula deduced by me is true.

H: $a_1 = \log_4 256 = \log_{4^1} 4^4 = \frac{\log 4^4}{\log 4^1} = \frac{4\log 4}{1\log 4} = \frac{4}{1} = 4$

$a_2 = \log_{16} 256 = \log_{4^2} 4^4 = \frac{\log 4^4}{\log 4^2} = \frac{4\log 4}{2\log 4} = \frac{4}{2} = 2$

$a_3 = \log_{64} 256 = \log_{4^3} 4^4 = \frac{\log 4^4}{\log 4^3} = \frac{4\log 4}{3\log 4} = \frac{4}{3}$

□ $a_3 = \log_{64} 256 = \frac{4}{3} = \frac{8}{6} = \frac{4\times 2}{4+2} = \frac{\log_4 256 \times \log_{16} 256}{\log_4 256 + \log_{16} 256}$

I: $a_1 = \log_6 7776 = \log_{6^1} 6^5 = \frac{\log 6^5}{\log 6^1} = \frac{5\log 6}{1\log 6} = \frac{5}{1} = 5$

$a_2 = \log_{36} 7776 = \log_{6^2} 6^5 = \frac{\log 6^5}{\log 6^2} = \frac{5\log 6}{2\log 6} = \frac{5}{2} = 2.5$

$a_3 = \log_{216} 7776 = \log_{6^3} 6^5 = \frac{\log 6^5}{\log 6^3} = \frac{5\log 6}{3\log 6} = \frac{5}{3}$

$\therefore a_3 = \log_{216} 7776 = \frac{5}{3} = \frac{12.5}{7.5} = \frac{5\times 2.5}{5+2.5} = \frac{\log_6 7776 \times \log_{36} 7776}{\log_6 7776 + \log_{36} 7776}$

Find the general statement that expresses $\log_{ab}x$

To find the general statement I will let $\log_a x = c$ and $\log_b x = d$. The conclusion withdrawn from this mathematical investigation indicate that numerator of the third answer is a product of first and second answers, when its denominator is a sum of first and second answer. As a logical continuation the general statement presents itself:

$$\log_{ab} x = \frac{\log_a x \times \log_b x}{\log_a x + \log_b x} = \frac{cd}{c+d}$$

Test the validity of your general statement using other values of *a, b* and *x*

The definition of logarithms provides us with the assumptions that *a, b, x > 0* and *a, b ≠ 1*. The best method to check the validity of my general statement is to use different values of *a, b* and *x*.

I) Let $a = 3$, $b = 9$ and $x = 729$

$$c = \log_a x = \log_3 729$$
$$d = \log_b x = \log_9 729$$
$$\log_{ab} x = \log_{27} 729$$

Next it is necessary to calculate to value of $\log_{27} 729 = \frac{\log 729}{\log 27} = 2$

Later the use of formula $\frac{c \times d}{c + d} = \frac{\log_3 729 \times \log_9 729}{\log_3 729 + \log_9 729} = \frac{6 \times 3}{6 + 3} = \frac{18}{9} = 2$

In this example the statement is true, because $a, b, x \in Q^+$

II) Let $a = 4$, $b = 6$ and $x = 1$

$$c = \log_a x = \log_4 1$$
$$d = \log_b x = \log_6 1$$
$$\log_{ab} x = \log_{24} 1$$

By the definition it is known that $\log_w q = j$, then $q = w^j$, therefore if q=1 then j = 0. In that case $\log_4 1 = 0$, $\log_6 1 = 0$ and $\log_{24} 1 = 0$. As a conclusion, the argument of a logartihm has to be different from 1.

Discuss the scope and limitations of *a, b* and *x*

For statement $\log_{ab} x = \dfrac{\log_a x \times \log_b x}{\log_a x + \log_b x}$ to be true two conditions have to be met:

1) $a, b, x > 0$

2) $a, b \neq 1$

Explain how you arrived at your general statement

I arrived at my general statement when I was thinking how to write the expression for n^th^ of each sequence in form $\dfrac{p}{q}$. Then I used the two formulas $\log_a b = \dfrac{\log_c b}{\log_c a}$ and $\log_a b^x = x \log_a b$. The series of calculations (and final one presented below) lead to the general statement.

$$\log_{ab} x = \frac{\log_x x}{\log_x ab} = \frac{1}{\log_x a + \log_x b} = \frac{1}{\dfrac{\log_a a}{\log_a x} + \dfrac{\log_b b}{\log_b x}} = \frac{1}{\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x}} = \frac{1}{\dfrac{\log_b x + \log_a x}{\log_a x \times \log_b x}} = \frac{\log_a x \times \log_b x}{\log_b x + \log_a x} = \frac{c \times d}{c + d}$$

Technology used

1. For all the calculations:
   CASIO GDC CFX-9850GB PLUS
2. For all the graphic presentation:
   Microsoft Office Word 2007
   MathType v. 6.5c