SL Portfolio Type 1

(Mathematical Investigation)

Infinite Surds

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This great work

is done by

Anh Linh, 5Y

I. Introduction:

In this portfolio, firstly I will find out the equation for the nth term of this infinite surd:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}}$$

And I will use data table and graph to suggest about the value of $a_n - a_{n+1}$ when it gets very large. After that I will prove an equation to calculate the exact value of the infinite surd. Followed by repeating all the same steps above but for the other infinite surd:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}}$$

After I repeated all the steps for this surd, I will consider about the general infinite surd:

$$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\dots}}}}}$$

Using all the steps that I did for the others two surd, I will find an expression for the exact value of this general infinite surd in terms of *k*. Furthermore, I will find the general statement that represents all the values of *k* for which the expression is an integer.

Nonetheless, I will discuss the scope and limitations of my general statement. And finally I will explain how I arrived my general statement and the integer expression.

II. BODY:

The following expression is an example of an infinite surd:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

Consider this surd as a sequence of term a_n where:

$$a_1 = \sqrt{1+\sqrt{1}}$$

$$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}}$$

$$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} \quad etc.$$

The formula for a_{n+1} in terms of a_n:

$$a_{n+1} = \sqrt{1+a_n}$$

Calculate the decimal values of the first ten terms of the sequence:

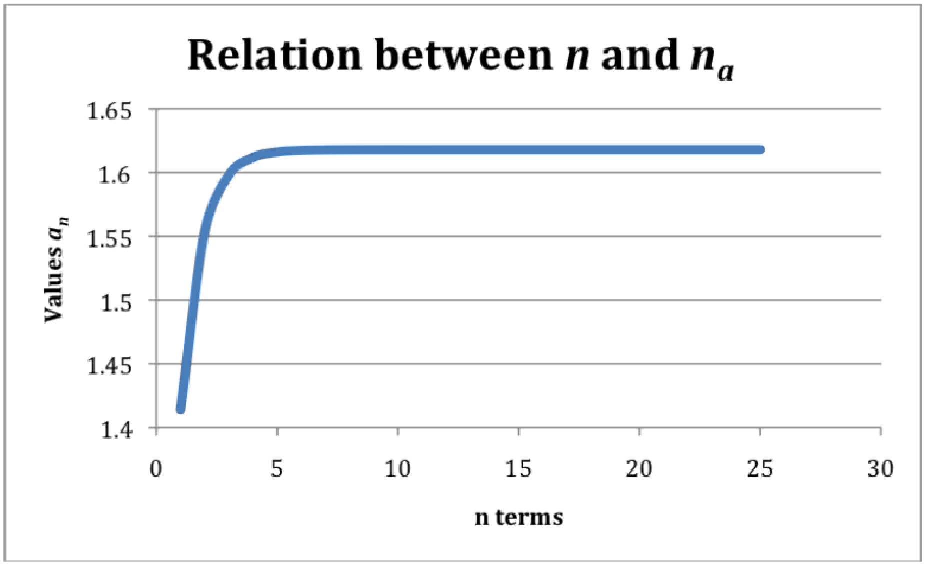
Using MS Excel 2008

Terms	Values
1	1.414213562
2	1.553773974
3	1.598053182
4	1.611847754
5	1.616121207
6	1.617442799
7	1.617851291
8	1.617977531
9	1.618016542
10	1.618028597
11	1.618032323
12	1.618033474
13	1.61803383
14	1.61803394
15	1.618033974
16	1.618033984
17	1.618033987
18	1.618033988
19	1.618033989
20	1.618033989
21	1.618033989
22	1.618033989
23	1.618033989
24	1.618033989
25	1.618033989

Notice:

The formula to calculate the second column: B2=SQRT(1+SQRT(1)) then following to the next values will be B3=SQRT(1+B2) etc…

Plot the relation graph between n and a_n:



(This graph I used the software Microsoft Excel 2008 to draw)

Evaluation:

As the graph shown, as the n gets very large, the values of a_n still be the same. Hence the value of $a_n - a_{n+1}$ is equal to 0. According to the data table, after the 4th term, all the data have the same until 1.61 which is 2 decimal place. And after a_{18}, all the values are exactly have the same value 1.618033989.

As we have:

$$a_{n+1} - a_n = \sqrt{1 + a_n} - a_n$$

However, when it comes to exact value:

$$\begin{aligned}
& a_{n+1} - a_n = 0 \\
& \Rightarrow \sqrt{1 + a_n} - a_n = 0 \\
& \Rightarrow \sqrt{1 + a_n} = a_n \\
& \Rightarrow 1 + a_n = a_n^{\,2} \\
& \Rightarrow a_n^{\,2} - a_n - 1 = 0
\end{aligned}$$

Using quadratic formula to solve this equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>Notice:</u> $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ This value is canceled out because the value of a square root can not be negative.

Thus:

$$a_n = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$a_n = \frac{1 + \sqrt{5}}{2}$$

$$\Rightarrow a_n = 1.6180$$

(correct)

Consider another infinite surd:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}}$$

The first term is $a_1 = \sqrt{2+\sqrt{2}}$

Calculate the decimal values of the first ten terms of this sequence:

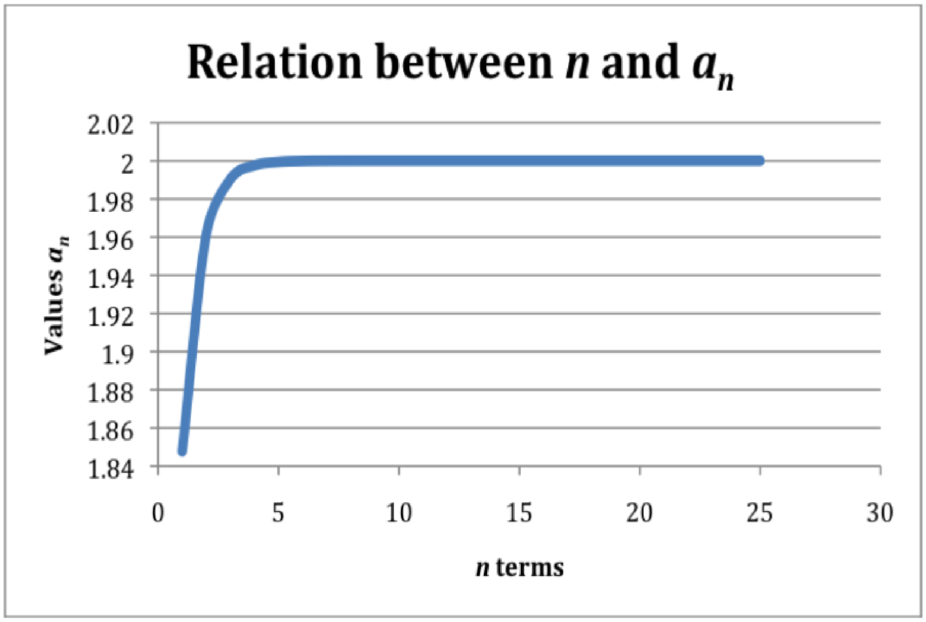
Using MS Excel 2008

Terms	Values
1	1.847759065
2	1.961570561
3	1.990369453
4	1.997590912
5	1.999397637
6	1.999849404
7	1.999962351
8	1.999990588
9	1.999997647
10	1.999999412
11	1.999999853
12	1.999999963
13	1.999999991
14	1.999999998
15	1.999999999
16	2
17	2
18	2
19	2
20	2
21	2
22	2
23	2
24	2
25	2

Notice:

The formula to calculate the second column: B2=SQRT(2+SQRT(2)) then following to the next values will be B3=SQRT(2+ B2) etc…

Plot the relation graph between n and a_n :



(This graph I used the software Microsoft Excel 2008 to draw)

<u>Evaluation:</u>

(I repeat the entire process above)

As the graph shown, as the *n* gets very large, the values of a_n still be the same. Hence the value of $a_n - a_{n+1}$ is equal to 0. According to the data table, the a_8, a_9 and a_{10} already have the same value 1.99999, equivalent until 5 decimal place.And after 15th term, all the value are exactly the same values 2.

As we have:

$$a_{n+1} - a_n = \sqrt{2 + a_n} - a_n$$

However, when it comes to exact value:

$$a_{n+1} - a_n = 0n$$
$$\Rightarrow \sqrt{2 + a_n} - a_n = 0$$
$$\Rightarrow \sqrt{2 + a_n} = a_n$$
$$\Rightarrow 2 + a_n = a_n^{\;2}$$
$$\Rightarrow a_n^{\;2} - a_n - 2 = 0$$

Using quadratic formula to solve this equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>Notice:</u> $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ This value is canceled out because the value of a square root can not be negative.

Thus:

$$a_n = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$
$$a_n = \frac{1 + \sqrt{5}}{2}$$
$$\Rightarrow a_n = 2$$
(correct)

Consider the general infinite surd:

$$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\ldots}}}}}$$

The first term is: $a_1 = \sqrt{k+\sqrt{k}}$

The a_{n+1} term in term of a_n:

$$a_{n+1} = \sqrt{k+a_n}$$

An expression for the exact value of this general infinite surd in terms of k:

$$a_{n+1} - a_n = \sqrt{k+a_n} - a_n$$
$$\Rightarrow a_{n+1} - a_n = 0n$$
$$\Rightarrow \sqrt{k+a_n} - a_n = 0$$
$$\Rightarrow \sqrt{k+a_n} = a_n$$
$$\Rightarrow k + a_n = a_n{}^2$$
$$\Rightarrow a_n{}^2 - a_n - k = 0$$

Using quadratic formula to solve this equation:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Thus:

$$a_n = \frac{-(-1)+\sqrt{(-1)^2-4(1)(-k)}}{2(1)}$$
$$\Rightarrow a_n = \frac{1+\sqrt{1+4k}}{2}$$

The exact value of this general infinite surd in terms of k is:

$$a_n = \frac{1+\sqrt{1+4k}}{2}$$

As we can see the value of an infinite surd $\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\dots}}}}}$ is not always an integer.

Example:

$k = 3$

$\Rightarrow \sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\dots}}}}}$

Table shows the first ten terms of this infinite surd:

(Using MS Excel 2008)

Terms	Values
1	2.175327747
2	2.274934669
3	2.296722593
4	2.301460969
5	2.302490167
6	2.302713653
7	2.302762179
8	2.302772715
9	2.302775003
10	2.3027755
11	2.302775608
12	2.302775631
13	2.302775636
14	2.302775637
15	2.302775638
16	2.302775638
17	2.302775638
18	2.302775638
19	2.302775638
20	2.302775638
21	2.302775638
22	2.302775638
23	2.302775638
24	2.302775638
25	2.302775638

Notice:

The formula to calculate the second column: B2=SQRT(3+SQRT(3)) then following to the next values will be B3=SQRT(3+ B2) etc…

Therefore, the value of an infinite surd is not always an integer.

To find some values of k that makes the expression an integer

$$a_n = \frac{1+\sqrt{1+4k}}{2}$$
$$\Rightarrow a_n \in Z$$

<u>1st method</u>

Hence, we will have this equation for the relation between a_n and k:

$$\Rightarrow a_n = \frac{1+\sqrt{1+4k}}{2}$$
$$2a_n = 1+\sqrt{1+4k}$$
$$2a_n - 1 = \sqrt{1+4k}$$
$$(2a_n - 1)^2 = 1+4k$$
$$4a_n{}^2 - 4a_n + 1 = 1+4k$$
$$\frac{4a_n{}^2 - 4a_n + 1 - 1}{4} = k$$
$$\Rightarrow a_n{}^2 - a_n = k$$
$$\Rightarrow k = a_n(a_n - 1)$$

Therefore the general statement for k is a product of two consecutive integers.

<u>2nd method</u>

Or I can use another method to prove this general statement and also check the result of it.

Back to the $a_n = \frac{1+\sqrt{1+4k}}{2}$, for a_n to be integer, $1+\sqrt{1+4k}$ must divided-able by 2 which are even number. Therefore, $\sqrt{1+4k}$ must be odd number. And $1+4k$ must be a perfect square, if not the result value will be irrational number.

Here are the table showing the first 50th term of perfect square:

$1^2 = 1$	$11^2 = 211$	$21^2 = 441$	$31^2 = 961$	$41^2 = 1681$
$2^2 = 4$	$12^2 = 144$	$22^2 = 484$	$32^2 = 1024$	$42^2 = 1764$
$3^2 = 9$	$13^2 = 169$	$23^2 = 529$	$33^2 = 1089$	$43^2 = 1849$
$4^2 = 16$	$14^2 = 196$	$24^2 = 576$	$34^2 = 1156$	$44^2 = 1936$
$5^2 = 25$	$15^2 = 225$	$25^2 = 625$	$35^2 = 1225$	$45^2 = 2025$
$6^2 = 36$	$16^2 = 256$	$26^2 = 676$	$36^2 = 1296$	$46^2 = 2116$
$7^2 = 49$	$17^2 = 289$	$27^2 = 729$	$37^2 = 1369$	$47^2 = 2209$
$8^2 = 64$	$18^2 = 324$	$28^2 = 784$	$38^2 = 1444$	$48^2 = 2304$
$9^2 = 81$	$19^2 = 361$	$29^2 = 841$	$39^2 = 1521$	$49^2 = 2401$
$10^2 = 100$	$20^2 = 400$	$30^2 = 900$	$40^2 = 1600$	$50^2 = 2500$

As we can see all the odd number perfect square will have an odd number as a result. Let call v^2 is the sum of $1+4k$ which v is an odd integer.

Then:

$$1+4k=v^2$$
$$\Rightarrow 4k=v^2-1 \qquad [1]$$
$$\Rightarrow k=\frac{v^2-1}{4}$$

Which v^2-1 is even number, and divided-able by 4. And v^2 is an odd number, $\Rightarrow v$ also an odd number

Formula of an odd number: $v=2u+1$ [2]

(u is a random integer number)

From [1] and [2] we have:

$$k=\frac{(2u+1)^2-1}{4}$$
$$\Rightarrow k=\frac{4u^2+4u+1-1}{4}$$
$$\Rightarrow k=u^2+u$$
$$\Rightarrow k=u(u+1)$$

Hence the general statement for k is a product of two consecutive integers.

Checking the general statement that represents all the values of k for which the expression is an integer using the 1st method.

If

$$a_n = 3$$
$$\Rightarrow k = 3(3-1) = 6$$

This infinite surd is $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\dots}}}}}$

<u>Table shows the first ten terms of this infinite surd:</u>

(Using MS Excel 2008)

Terms	Values
1	2.906800603
2	2.984426344
3	2.997403267
4	2.99956718
5	2.999927862
6	2.999987977
7	2.999997996
8	2.999999666
9	2.999999944
10	2.999999991
11	2.999999998
12	3
13	3
14	3
15	3
16	3
17	3
18	3
19	3
20	3
21	3
22	3
23	3
24	3
25	3

<u>Notice:</u>

The formula to calculate the second column: B2=SQRT(6+SQRT(6)) then following to the next values will be B3=SQRT(6+B2) etc…

After checking the values table of the infinite surd $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\dots}}}}}$ and $a_n = 3$. Hence, the general statement that represents all the values of k for which the expression is an integer is correct.

Now I will test the 2nd method using a greater number, for example 99

If

$u = 99$

$\Rightarrow k = 99\,(99\ +1) = 9900$

This infinite surd is $\sqrt{9900\ +\sqrt{9900\ +\sqrt{9900\ +\sqrt{9900\ +\sqrt{9900\ +\dots}}}}}$

Table shows the first ten terms of this infinite surd:

(Using MS Excel 2008)

Terms	Values
1	99.99749369
2	99.99998747
3	99.99999994
4	100
5	100
6	100
7	100
8	100
9	100
10	100
11	100
12	100
13	100
14	100
15	100
16	100
17	100
18	100
19	100
20	100
21	100
22	100
23	100
24	100
25	100

Notice:

The formula to calculate the second column: B2=SQRT(9900+SQRT(9900)) then following to the next values will be B3=SQRT(9900+B2) etc…

Therefore, the 2nd method also correct as which the exact value is 100, an even and divided-able by 2 integer.

III. Conclusion:

After all the tests and checking the general statement, they prove that my result are true and correct for all the values of *k*. Nonetheless, I have also solved the other questions at the beginning of this portfolio and found the general formula for the exact value of this $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}}$ and this $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}}$ infinite surds in particular and this $\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\dots}}}}}$ surd formula in general with difference method and examples. As this part, I have done all the question of the mathematics portfolio for IB students 'Infinite Surd'/