IB Mathematics SL
Portfolio Type I

Matrix Powers

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IB II

- Introduction:

Matrices are rectangular tables of numbers or any algebraic quantities that can be added or multiplied in a specific arrangement . A matrix is a block of numbers that consists of columns and rows used to represent raw data, store information and to perform certain mathematical operations . The aim of this portfolio is to find general formulas for matrices in the form of $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.
Each set of matrices will have a trend in which a general formula for each example is deduced.

- Method 1:

Consider the matrix M = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ when k = 1.

Table 1: Represents the trend in matrix M = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ as n is changed in each trial.

Power	Matrix
n = 1	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
n = 2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
n = 3	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$
n = 4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$
n = 5	$\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$
n = 10	$\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$
n = 20	$\begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$

Matrix M is a 2 x 2 square matrix which have an identity. As n changes the zero patterns is not affected while the 2 is affected. $2^{\mathbf{n}}$ is raised to the power of n. When n =1, $2^1 = 2$, when n = 2, $2^2 = 4$ and when n = 3, $2^3 = 8$ and so on. So as a conclusion, $M^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$

- Method 2

Consider the matrices $P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $S = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

Table 2.1: Represents matrix P as the power n is increased by one for each trial.

Power	Matrix / P^n
n =1	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^1 = 2^0 \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$
n =2	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2 = 2^1 \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$
n = 3	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^3 = 2^2 \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$
n = 4	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^4 = 2^3 \begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix} = \begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$
n = 5	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^5 = 2^4 \begin{pmatrix} 33 & 31 \\ 31 & 33 \end{pmatrix} = \begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix}$

As the power n of the matrix is increased by one the scalar is doubled. To find the elements inside the matrix, the scalar that is used in the trial is doubled and then added to the elements in the matrix. E.g.: when n = 4, the scalar is 8 (4x2=8) and then this amount, 8, is added to the elements inside of the matrix for n=3 (9+8=15). So the general formula deduced is $P^n = n \begin{pmatrix} x+2 & y+2 \\ y+2 & x+2 \end{pmatrix}$ where n=2. Also, x and y represent the elements of the previous matrix with a power of n – 1. Note that this formula needs two consecutive matrix powers in order to be applied. Another general formula for this trend can be found. As the power n changes, the value to which the scalar is raised changes. $P^n = 2^{n-1} \begin{pmatrix} 2^n-1 & 2^n+1 \\ 2^n+1 & 2^n-1 \end{pmatrix}$ n is an integer.

To make sure of the validity of the formula n = 3 is used as an example.

Using GDC: $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^3 = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$

Using the general formula: $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^3 = 2^2 \begin{pmatrix} 2^n-1 & 2^n+1 \\ 2^n+1 & 2^n-1 \end{pmatrix} = 4 \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$

Table 2.2: Represents matrix S as the power n is increased by one for each trial.

Power	Matrix / S^n
n =1	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^1 = 2^0 \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
n = 2	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^2 = 2^1 \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$
n = 3	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^3 = 2^2 \begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix} = \begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix}$
n = 4	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^4 = 2^3 \begin{pmatrix} 82 & 80 \\ 80 & 82 \end{pmatrix} = \begin{pmatrix} 656 & 640 \\ 640 & 656 \end{pmatrix}$
n = 5	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^5 = 2^4 \begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix} = \begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix}$

As for two consecutive matrices, the trend is found so as the power n increases by a factor of one. The scalar is doubled in each trial and then a certain factor is added to the elements of matrix S. taking n = 2 as an example, the scalar is found by doubling the power n = 1 (1x2=2). The factor of addition is determined by multiplying the difference in x and y inside the matrix with a power less with one by 3. In matrix n = 1, the difference in the elements is 2 (4-2=2), the answer is multiplied by 3 (2x3=6). Finally 6 is added to the matrix of n = 2. The general formula for this trend is $S^{\mathbf{n}} = 2 \begin{pmatrix} x+3 & y+3 \\ y+3 & x+3 \end{pmatrix}$ where n = 1. Notice that this formula needs two consecutive matrix powers in order to be applied. Another general formula can be derived for this sequence. As the power n changes, the power in which the scalar is raised will change also.

$S^{\mathbf{n}} = 2^{\mathbf{n-1}} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$ where n is an integer. To check the validity of this formula n = 4 is used as an example.

Using GDC: $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^4 = \begin{pmatrix} 656 & 640 \\ 640 & 656 \end{pmatrix}$

Using the General Formula: $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^4 = 2^3 \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix} = 8 \begin{pmatrix} 82 & 80 \\ 80 & 82 \end{pmatrix} = \begin{pmatrix} 656 & 640 \\ 640 & 656 \end{pmatrix}$.

- Method 3:

1. Consider the matrices in the form Q = $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.

Table 3: Represents the trend in matrix Q as k is increased by one in each trial.

Power	Matrix	General Formula
k = 1	M = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$M^n = 2^{n-1} \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$
k = 2	P = $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	$P^n = 2^{n-1} \begin{pmatrix} 2^n-1 & 2^n+1 \\ 2^n+1 & 2^n-1 \end{pmatrix}$
k = 3	S = $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$	$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$
k = 4	D = $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$	$D^n = 2^{n-1} \begin{pmatrix} 4^n+1 & 4^n-1 \\ 4^n-1 & 4^n+1 \end{pmatrix}$
k = 5	F = $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$	$F^n = 2^{n-1} \begin{pmatrix} 5^n+1 & 5^n-1 \\ 5^n-1 & 5^n+1 \end{pmatrix}$
k = 6	N = $\begin{pmatrix} 7 & 5 \\ 5 & 7 \end{pmatrix}$	$S^n = 2^{n-1} \begin{pmatrix} 6^n+1 & 6^n-1 \\ 6^n-1 & 6^n+1 \end{pmatrix}$

The general formulas of matrix M (k = 1), matrix P (k = 2), and matrix S (k = 3) were established by giving evidence in the above methods. Following a certain trend, as k is changed (starting from k = 2), the elements of the general formula in which n is raised to it changed also. The scalar is kept constant for all different values of k. So as a conclusion, rule A states that $Q^{\mathbf{n}} = 2^{\mathbf{n-1}} \begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$.

To check this formula, matrix D is used where k = 4 and n = 5

Using GDC: $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}^5 = \begin{pmatrix} 16400 & 16368 \\ 16368 & 16400 \end{pmatrix}$

Using rule A: $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}^5 = 2^4 \begin{pmatrix} 4^5+1 & 4^5-1 \\ 4^5-1 & 4^5+1 \end{pmatrix} = 16 \begin{pmatrix} 1025 & 1023 \\ 1023 & 1025 \end{pmatrix} = \begin{pmatrix} 16400 & 16368 \\ 16368 & 16400 \end{pmatrix}$

Another trend is recognized for the consecutive matrices, as k is increased by 1 each element of the new matrix is increased by one too. In other words, square matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is added to the previous matrix in order to obtain the result. A general formula can be established, which is:

$Q^{\mathbf{n}}$ = Matrix R + Matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, where R is matrix of k – 1. To verify this formula matrix F (k = 5) and N (k = 6) are going to be taken as examples.

matrix X = Matrix F + Matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, are matrices X and N equal??

Matrix N = $\begin{pmatrix} 7 & 5 \\ 5 & 7 \end{pmatrix}$

Matrix X = $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 5 & 7 \end{pmatrix}$, so this statement is true.

- ## Method 4:

In this method, other values such as negative integers, fraction numbers and irrational values are going to be investigated as the values of k. those values are going to be applied on rule A, this will emphasize any limitations, if found, for this rule.

Rule A states that $Q^{\mathbf{n}} = 2^{\mathbf{n-1}} \begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$, the following trials are tried:

1. **<u>Negative Values:</u>**

Matrix B represents the matrix where k is a negative integer raised to the power of n, k = -3 and n = 2, in a matrix of the form $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^{n}$, so matrix B = $\begin{pmatrix} -2 & -4 \\ -4 & -2 \end{pmatrix}$. From rule A and the above examples, the following formula is deduced: $B^{\mathbf{n}} = 2^{\mathbf{n-1}} \begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$

Using GDC: $\begin{pmatrix} -2 & -4 \\ -4 & -2 \end{pmatrix}^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$

Using Rule A: $\begin{pmatrix} -2 & -4 \\ -4 & -2 \end{pmatrix}^2 = 2^{\mathbf{2-1}} \begin{pmatrix} -3^2+1 & -3^2-1 \\ -3^2-1 & -3^2+1 \end{pmatrix} = 2 \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$

As a conclusion, rule A also can be applied when k is a negative value.

2. Fraction Values:

Matrix L represents the matrix where k is a fraction value raised to the power of n, k = 3/2 and n = 3 in a matrix of the form $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^{n}$, so matrix L = $\begin{pmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{pmatrix}$. From the above examples, $L^{\mathbf{n}} = 2^{\mathbf{n-1}} \begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$.

Using GDC: $\begin{pmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{pmatrix}^3 = \begin{pmatrix} 35/2 & 19/2 \\ 19/2 & 35/2 \end{pmatrix}$

Using Rule A: $\begin{pmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{pmatrix}^3 = 2^{3-1} \begin{pmatrix} (3/2)^3+1 & (3/2)^3-1 \\ (3/2)^3-1 & (3/2)^3+1 \end{pmatrix} = 4 \begin{pmatrix} 35/8 & 19/8 \\ 19/8 & 35/8 \end{pmatrix}$
$= \begin{pmatrix} 35/2 & 19/2 \\ 19/2 & 35/2 \end{pmatrix}$.

From the above results, it is shown that rule A is valid even for fraction numbers.

3. Irrational Values:

Matrix U represents the matrix where k is an irrational number which is raised to the power of n, k = $\prod$ and n = 2 in a matrix of the form $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^{n}$, so matrix L = $\begin{pmatrix} \prod+1 & \prod-1 \\ \prod-1 & \prod+1 \end{pmatrix}$. From the above examples, $U^{\mathbf{n}} = 2^{\mathbf{n-1}} \begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$.

Using GDC: $\begin{pmatrix} \prod+1 & \prod-1 \\ \prod-1 & \prod+1 \end{pmatrix}^2 = \begin{pmatrix} 21.739208 & 17.739208 \\ 17.739208 & 21.739208 \end{pmatrix}$.

Using Rule A: $\begin{pmatrix} \prod+1 & \prod-1 \\ \prod-1 & \prod+1 \end{pmatrix}^2 = 2^{2-1} \begin{pmatrix} \prod^2+1 & \prod^2-1 \\ \prod^2-1 & \prod^2+1 \end{pmatrix} =$
$2 \begin{pmatrix} 10.869604 & 8.869604401 \\ 8.869604401 & 10.869604 \end{pmatrix} = \begin{pmatrix} 21.739208 & 17.739208 \\ 17.739208 & 21.739208 \end{pmatrix}$

From those final results, we can prove that this statement, Rule A, can also be applied to irrational integers.