Math Portfolio

Type 1 Task

Investigating areas and volumes

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Introduction

In this portfolio, I'm going to investigate ratios of areas and volumes of power func tions, which can be generalized as the following:

$Y=x^n$, where n-is the power, $n\in R$

This function will be graphed between two arbitrary parameters *x=a* and *x=b* such that *a<b* (in other words boundaries or limits). In this investigation, I'm going to use method of math induction, integration using power rule, application of integration (areas under the curve and solids of revolution), also some knowledge about power functions.

Investigation Process

Given the power function $y=x^2$, graph of which is parabola. I need to consider the region formed by this function from x=0 to x=1 and the x-axis, let's label this area B; and the region from y=0 to y=1 and the y-axis area A. This can be shown on the illustration (Fig.1) below:

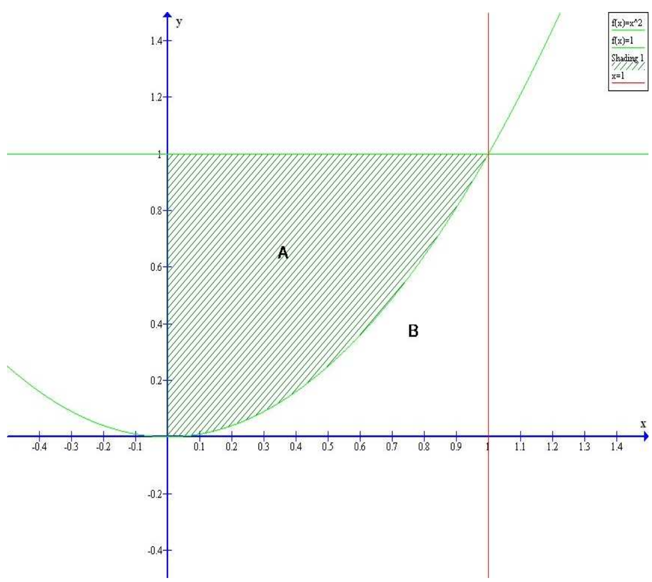
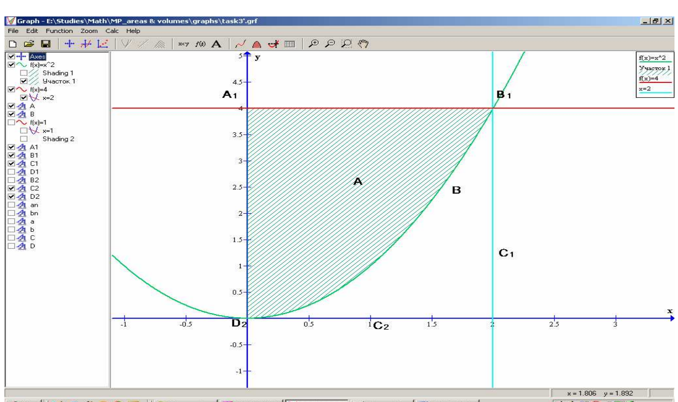


Figure 1

To plot this, I used *Graph 4-3 Software*, which I found very convenient.



So the areas formed and illustrated above (Figure 1), need to be considered in order to find a ratio between them, hence *area A: area B*. To find that, I need to use integration using power rule. Integration, using power rule is the general integration rule, which is:

$\int_a^b ax^n dx = \frac{ax^{n+1}}{n+1}\Big|_a^b$ where n≠-1

It consists of the following steps:

1. Calculate area B, which is $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$
2. Calculate area of the quadrate (formed by the x and y boundaries of the function), which is $\int_0^1 1dx = x\Big|_0^1 = 1 - 0 = 1$
3. Calculate area A, which is equal to
   area A = (area of the quadrate – area B)

   which is $S_A = 1 - \frac{1}{3} = \frac{2}{3}$
4. Step4. Therefore the ratio of the areas, will be
   $S_A : S_B = \frac{2}{3} : \frac{1}{3} = 2 : 1$

These steps, I repeated for other power functions of the type $y=x^n$, where n is the power, which is the set of the positive integers between x=0 and x=1. The graphs can be seen on the Figure 2

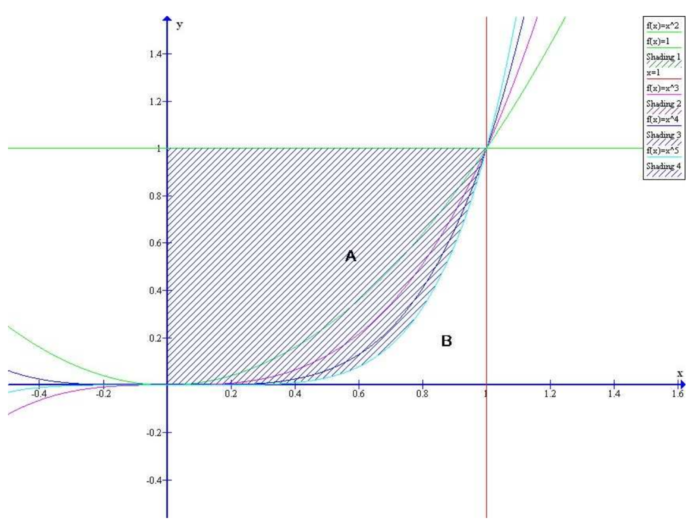


Figure 2*

* - this function is considered with the following boundaries x=0 and x=1 and x -axis; y=0 and y=1 and y-axis

On the figure above, I wanted to show you four functions, which were $y=x^2$, $y=x^3$, $y=x^4$, and $y=x^5$. The functions are labeled and the legend is shown in the right corner. The results, I want present you on the following table:

Function	Area A	Area B	Ratio
$Y= x^2$	0.66	0.34	2 : 1
$y=x^3$	0.75	0.25	3 : 1
$y=x^4$	0.80	0.20	4 : 1
$y=x^5$	0.83	0.17	5 : 1

Tab 1

From the table (Tab 1) I can see that, with change of the power of the function, the ratio of the areas changes as well. I would like also to say that, when n is a positive integer, the graph will look as the following:

n >0	
n-even	n-odd

Tab 2

From this observation, I can make a certain conjecture, which is

$\therefore$ If the power function of the type $y=x^n$, where n – is the power, $n \in Z^+$ and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to ***n: 1.***

However, notice that my conjecture works or was tested only for the set of positive integers ($n \in Z^+$). Now, I need to test my conjecture for other subsets of the real numbers, such as set of integers Z; set of rational numbers Q; set of positive rational numbers Q^+; and set of positive real numb ers R and irrational numbers.

So, first of all, I will try subset of the rational numbers, - negative integers (I will not test positive integers, because I already done that previously). So, let's try different negative values of *n* – power of the function, and we will see if it works or not. I will not show calculations for that (I'm using same method of power integration rule); I will just right the results and presen t them on the table below (Tab 3).

Function	Area A	Area B	Ratio
$Y=x^{-2}$	0.34	0.66	1: 2
$Y=x^{-3}$	0.25	0.75	1: 3
$Y=x^{-4}$	0.20	0.80	1: 4
$Y=x^{-5}$	0.17	0.83	1: 5
$Y=x^{-6}$	0.14	0.86	1: 6
$Y=x^{-7}$	0.13	0.87	1: 7

Tab 3

The table above represents the values of the areas and their ratios (S_A: S_B). In order to see it clearly, I can show you a chart wit h functions presented there.

n<0	
n-even	n-odd

Tab 4

I would like to pay attention, on that I'm ignoring the negative sign, when I'm doing a ratio; therefore I would take it as an absolute value - $|n|$. Therefore my conjecture for the set of negative integers will be as the following:

$\therefore$ *If the power function of the type y=x^n, where n – is the power, n$\in$Z (where* $Z = \{0,\pm1,\pm2,\pm3,\ldots\}$ *and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to* ***1: |n|.***

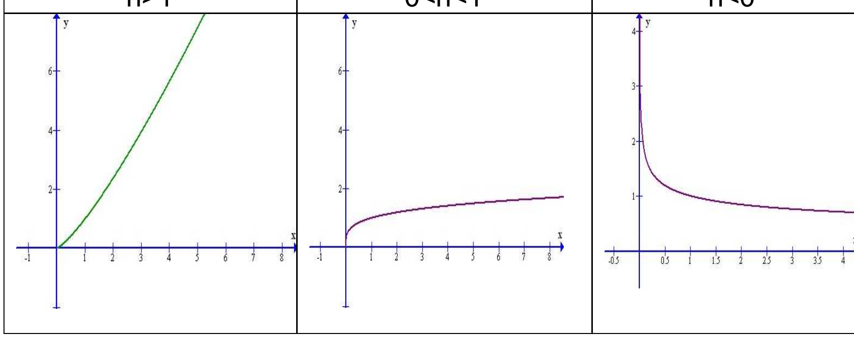
Next subset of rational numbers will be fractions, so will consider following functions represented on the following table:

Function	Area A	Area B	Ratio
Y=$x^{\frac{1}{2}}$	0.34	0.66	1: 2
Y=$x^{\frac{1}{3}}$	0.25	0.75	1: 3
Y=$x^{\frac{1}{4}}$	0.20	0.80	1: 4
Y=$x^{\frac{1}{5}}$	0.17	0.83	1: 5
Y=$x^{\frac{1}{6}}$	0.14	0.86	1: 6
Y=$x^{\frac{1}{7}}$	0.13	0.87	1: 7

Tab 5

So, the sketches to the functions will be as the followings:

n-rational number (fraction)		
n>1	0<n<1	n<0



Tab 6

The general conjecture, for the subset of fractions, will be:

$\therefore$ If the power function of the type y=x^n, where n – is the power, $n \in Q$ (where Q=$\left\{ x \middle| x = \frac{a}{b}, b \neq 0 \right.$ and a and b are integers} and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to 1:$\frac{1}{|n|}$.

To generalize, these examples, I would like to show a graph (Figure 3) Below, this will show, the comparison of the graphs by their powers:

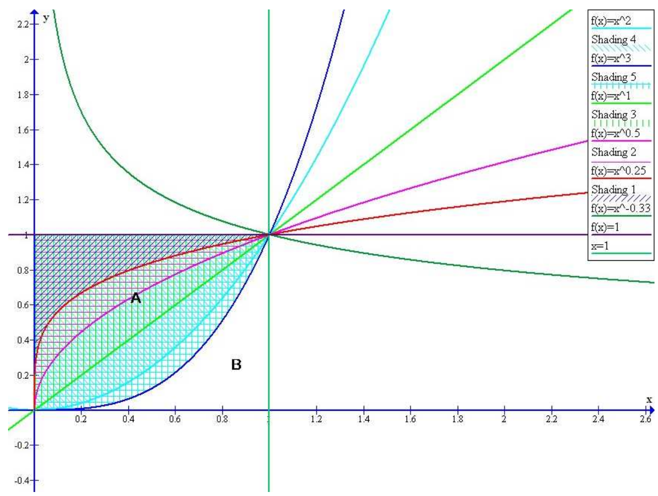


Figure 3

Regardless to the graph and the previous observations for the different subsets of the real numbers, conjecture might be slightly changed. Also note, that I didn't investigate irrational numbers such as π. So conjecture will be like:

$\therefore$ If the power function of the type $y=x^n$, where n – is the power,

- *where n – is the power, $n \in Z^+$ and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to **n: 1.***
- *where n – is the power, $n \in Z$ (where $Z = \{0, \pm1, \pm2, \pm3, \ldots\}$ and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to **1: |n|.***
- *where n – is the power, $n \in Q$ (where Q=$\left\{ x \middle| x = \frac{a}{b}, b \neq 0 \right.$ and a and b are integers} and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to **1:$\frac{1}{|n|}$.***

These conjectures are true for the boundaries, such as x=0 and x=1. I think I need to examine this for the x=0 and x=2; x=1 and x=2. So let's consider the same function $y=x^2$, but for different limits (Figure 4 and Figure 5)

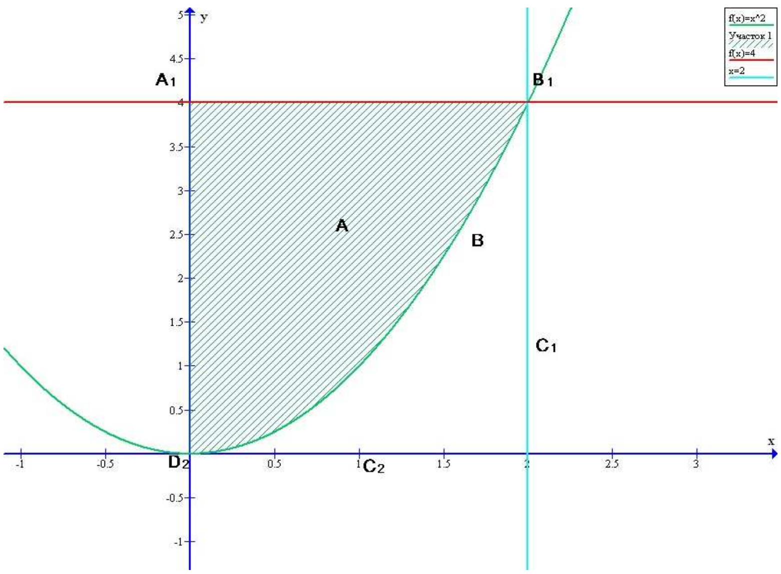


Figure 4

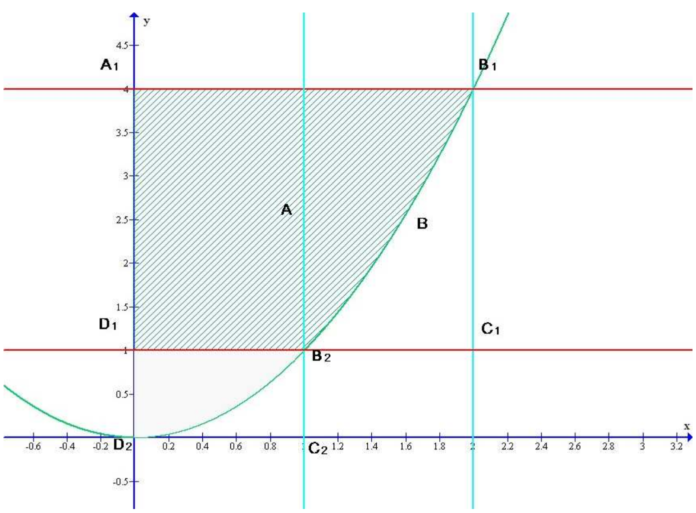


Figure 5

So, now limits (x=1 and x=2) are changed* and the result was done using the same method:

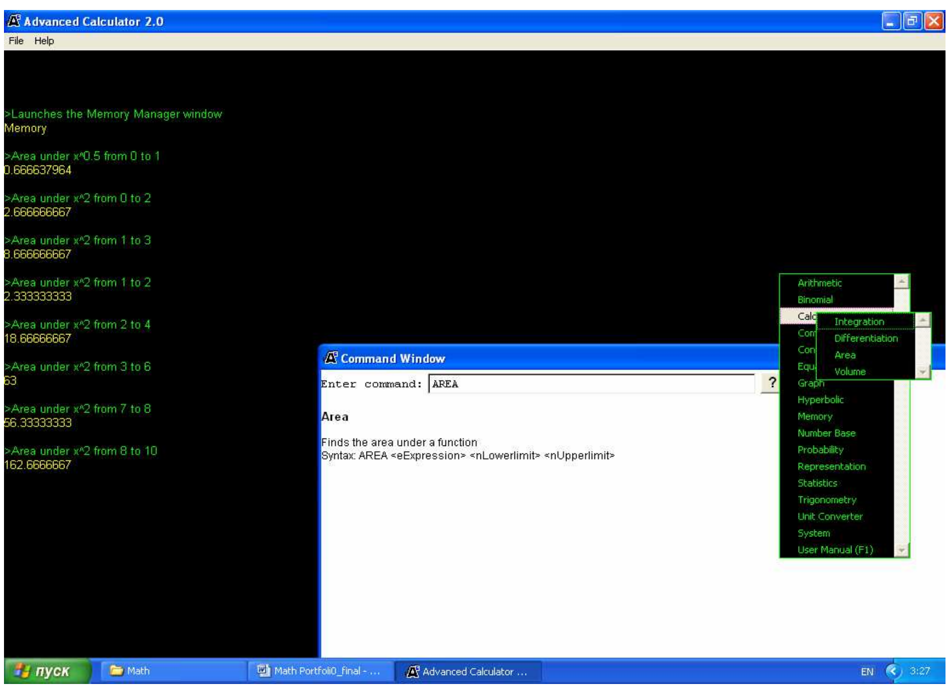
1. $S_B = \int_1^2 x^2 dx = 2.33$
2. S_A = 8-1-2.33=4.67
3. Ratio will be area A : area B=4.67 : 2.33=2 : 1

Or, to calculate the same thing I can use *Advanced Calculator 2.0. Computer Software*, it's easy to operate and less time consuming. In order to see a certain pattern, if there is so, let's take different boundaries, and calculate the area under the curve of the function $y=x^2$ using *Advanced Calculator Software*:

Boundaries	Area A	Area B	Ratio
X=0 and x=2	4.34	2.66	2 : 1
X=1 and x=2	4.34	2.66	2 : 1
X=1 and x=3	17.34	8.66	2 : 1
X=2 and x=4	37.34	18.66	2 : 1
X=3 and x=6	126	63.00	2 : 1
X=7 and x=8	112.67	56.33	2 : 1
X=8 and x=10	325.34	162.66	2 : 1

Tab. 3.0.

From the table above, I can say that, the conjecture is hold for different limits (positive values of x are only taken). So, limits don't affect the entire conjecture, there is no pattern seen. However, I would like to mention that with change of the domain (x - values, limits) range (y-value) changes respectively. In order to find area using other way (not manual one) I used *Advanced Calculator 2.0. Computer Software.*



*** - limits, are changed, therefore now area A will be equal to (area of the big rectangular $A_1B_1C_1D_1$ – area of the small rectangular $D_1B_2C_2D_2$ - area B)**

Area under the curve is an application of integration, so there should be some general formulae for it, so let's consider a power function of the type $y=x^n$ where $n \in R$ from x=a to x=b such that a<b and for the regions defined below:

Area A: $y=x^n$, $y=a^n$, $y=b^n$ and the y-axis

Area B: $y=x^n$, x=a, x=b and the x-axis

For better understanding, I will use the following diagram (Figure 6):

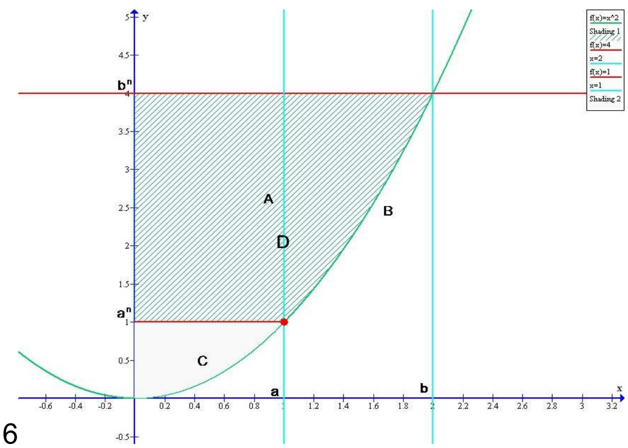


Figure 6

On the diagram there is a graph of power function $y=x^n$, with the boundaries x=a and x=b; $y=a^n$ and $y=b^n$. And I need to find a general formula for the ratio of the areas A and B.

Now, I will try to derive a general formula for the area B:

$$S_B = \int_a^b x^n dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} = \frac{b^{n+1}-a^{n+1}}{n+1} \text{ (i)}$$

Then area A, but I also would like to explain generally, how I'm going to find it; first of all, I need to consider area of the small rectangular, lets say area C ; then the whole area A will be equal to (area D-area C- area B).

$$S_A = S_D - S_C - S_B \text{ (ii)}$$

$$S_D = b * b^n$$

$S_C = a * a^n$ Because S=ab, where a and b-sides of the rectangular

Therefore just substitute the values of these areas into formula (ii)

$$S_A = |b*b^n| - (a*a^n) - \left(\frac{b^{n+1}-a^{n+1}}{n+1}\right) = (b^{n+1}-a^{n+1}) - \left(\frac{b^{n+1}-a^{n+1}}{n+1}\right) = |b^{n+1}-a^{n+1}|\left(1-\frac{1}{n+1}\right) =$$

$$|b^{n+1}-a^{n+1}|\left(\frac{n}{n+1}\right)$$

And, final result, the ratio of the areas:

$$S_A : S_B = \left| b^{n+1} - a^{n+1} \right| \left(\frac{n}{n+1} \right) : \left(\frac{b^{n+1} - a^{n+1}}{n+1} \right) = n : 1$$

S_A : S_B =n: 1

I can prove that using method of math induction:

Let P (n) be the proposition that S_A : S_B=n: 1

1. Test for n=1

S_B=

S_A=

S_A : S_B=

2. Let P (n) be true for n=k, i.e.,

S_B=

S_A=

S_A : S_B=

3. Test for n=k+1

S_B=

S_A=

S_A : S_B=

Thus, if the proposition is true for n=k then its true for n=k+1 as proved. As it true for n=1, then it must be true for n=1+1(n=2). As it true for n=2 then it must hold for n=2+1(n=3) and so on for the set of real numbers.

That is, by the principal of mathematical induction P (n) is true.

So, my general conjecture for the area will be as the following:

$\therefore$ *If the power functions of the type $y=x^n$, where n – is the power, $n \in R$ and has limits such as x=a and x=b and x-axis; and $y=x^n$, x=a, x=b and the x-axis. The ratio of the areas formed by the function and these boundaries will be equal to* ***n: 1.***

Also, I need to derive general formulae for the volumes of revolution generated by the region A and B when they are each rotated about

a) the x-axis
b) the y-axis

a) For the volumes of revolution, those are rotating around the x -axis, the following will be true:

$$V = \pi \int_{x=a}^{x=b} \left| f(x) \right|^2 dx$$

(iii) where x=a and x=b are boundaries, such as f(x) its a given function, particularly in that case it's a power function $f(x)=x^n$

This can be seen on the Figure 6 , which is plotted below

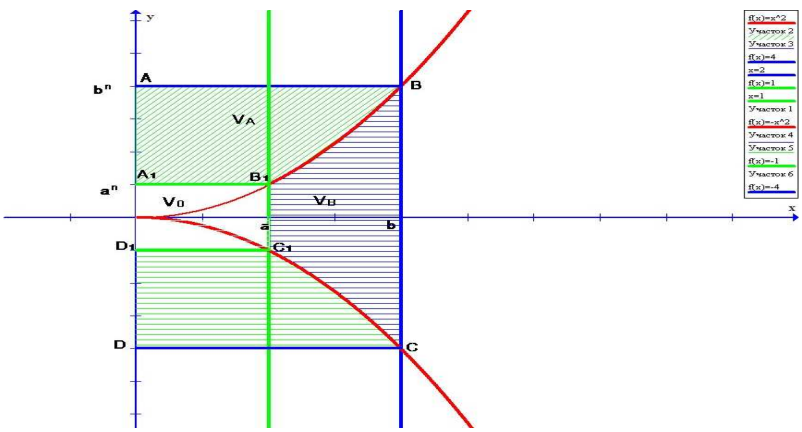


Figure 6

This graph shows two volumes (A and B); region, which is colored in blue, is V_B and region, colored in green is V_A. I need to find a ratio of those. So, I will try to derive a general formula for volumes of revolution rotated around x -axis.

According, to the formulae (iii), volume B will be equal to:

$$V_B = \pi \int_a^b x^{2n} dx \Big|_a^b = \pi \left(\frac{b^{2n+1} - a^{2n+1}}{2n+1} \right) \qquad \text{(iv)}$$

In order to determine V_A; I need to consider volume of the cylinder, because when it's rotating around x-axis, it creates a cylinder. The volume of the cylinder (ABCD) equals

$V_C = \pi R^2 H$, where R is b^n and height is b

And also, I need to consider V_0 that is another small cylinder ($A_1B_1C_1D_1$) which creates something like a hole inside the rotating figure, which is

$V_0 = \pi R^2 H$, where R is a^n and height is a

So, therefore $V_A = V_C - V_B - V_0$

$$V_A = \pi(b^{2n} * b) - \pi\left(\frac{b^{2n+1} - a^{2n+1}}{2n+1}\right) - \pi(a^{2n} * a) = \pi\left[b^{2n+1} - a^{2n+1} \left(1 - \frac{1}{2n+1}\right)\right] =$$ (v)

$$\pi\, b^{2n+1} - a^{2n+1} \left(\frac{2n}{2n+1}\right)$$

$$V_A : V_B = \pi\, b^{2n+1} - a^{2n+1} \left(\frac{2n}{n+1}\right) : \pi\left(\frac{b^{2n+1} - a^{2n+1}}{2n+1}\right) = 2n : 1$$

So, the general formula for this is:

$V_A : V_B = 2n : 1$

I can prove this formula, using method of math ind uction:

Let P (n) be the proposition that $V_A : V_B = 2n : 1$

1. Test for n=1

$V_B=$

$V_A=$

$V_A : V_B=$

2. Let P (n) be true for n=k, i.e.,

$V_B=$

$V_A=$

$V_A : V_B=$

3. Test for n=k+1

$V_B=$

$V_A=$

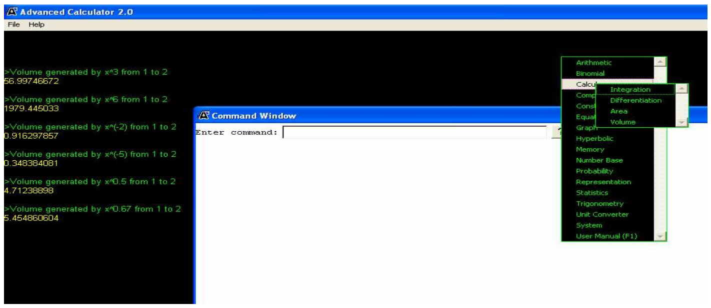
$V_A : V_B=$

Thus, if the proposition is true for n=k then its true for n=k+1 as p roved. As it true for n=1, then it must be true for n=1+1(n=2). As it true for n=2 then it must hold for n=2+1(n=3) and so on for the set of real numbers.

That is, by the principal of mathematical induction P (n) is true.

Hence my conjecture for the solids of revolution around x-axis will be as the following:
∴ If the power functions of the type $y=x^n$, where n – is the power, $n \in R$ and has limits such as x=a and x=b and x-axis. The ratio of the volumes formed by the function and these boundaries will be equal to ***2|n|: 1.***

In order to see, I would like to present some examples, which were found using *Advanced Calculator 2.0. Software*



Note, that the boundary is from x=1 to x=2; and it is rotated around x-axis. So, the results were as the following:

Function	V_A	V_B	Ratio
$Y=x^3$	342.00	57.00	**6 : 1**
$Y=x^6$	23753.40	1979.45	**12 : 1**
$Y=x^{-2}$	-3.68	0.92	**4 : 1**
$Y=x^{-5}$	-3.50	0.35	**10 : 1**
$Y=x^{0.5}$	4.71	4.71	**1 : 1**
$Y=x^{0.67}$	7.30	5.45	**1 : 1**

Tab. 3.0.

This table only proves that the conjecture works, for $n \in Z$, Q, and R.

b) For the volumes, which are rotating around y-axis, the following will be true:

$\boldsymbol{V = \pi \int [f^{-1}(y)]^2 dy}$, where $y=a^n$ and $y=b^n$ are the boundaries of the given function y=f(x)

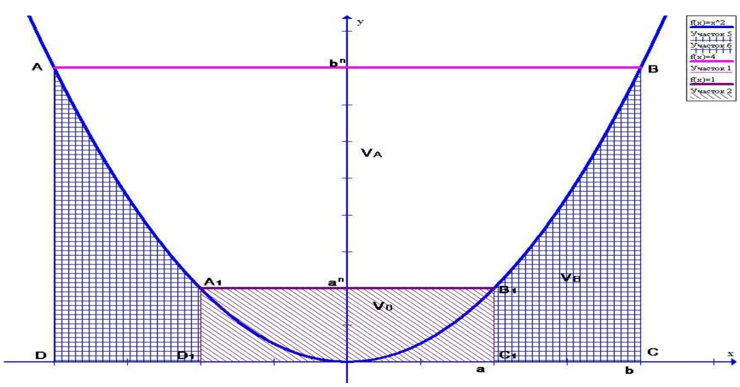


Fig. 7.0.

On the figure above, there is a rotation around y -axis. Let say that the area sh aded in blue is a V_B and just white region is V_A. As far as I need to integrate an inverse of the given function, I would like to show the process itself; because it might cause a mistake.

1. Given, f (x) = x^n
2. Need to find, $f^{-1}(x)$; which is $y=x^n$

$x=\sqrt[n]{y}$

I will try to derive a formula for solids of revolution around y-axis.

$$V_A = \pi\int_{a^n}^{b^n}(y^{\frac{1}{n}})^2 dy = \frac{y^{\frac{2}{n}+1}}{\frac{2}{n}+1}\Big|_{a^n}^{b^n} = \pi n\frac{y^{\frac{2+n}{n}}}{2+n}\Big|_{a^n}^{b^n} = \pi n\frac{b^{n+2}-a^{n+2}}{n+2} \quad \text{(vi)}$$

In order to find V_B; I need to consider volume of the cylinder created by

$V_C = \pi R^2 H$, where R is b and height is b^n

And ***$V_0=\pi R^2 H$, where R is a and height is a^n***

So, therefore, $V_B=V_C-V_A-V_0$ where V_C is a volume of the big cylinder (ABCD) and V_0 is a volume of the small cylinder inside ($A_1B_1C_1D_1$)

$$V_B=\pi b^{2+n}-\pi a^{2+n}-\pi n\frac{(b^{n+2}-a^{n+2})}{n+2}=\pi\ b^{2+n}-a^{2+n}\left(1-\frac{n}{n+2}\right)=\pi\ b^{n+2}-a^{n+2}\left(\frac{2+n-n}{n+2}\right) \quad \text{(vii)}$$

Then, I plug in the derived formulas (vi) and (vii) the ratio of the volumes will be:

$$V_A : V_B = \pi n\frac{b^{n+2}-a^{n+2}}{n+2} : 2\pi\frac{b^{n+2}-a^{n+2}}{n+2} = |n| : 2$$

I can prove this formula, using method of math induction (as it was shown previously) Therefore, I can state my conjecture as the following:

$\therefore$ *If the power functions of the type $y=x^n$, where n – is the power, $n \in R$ and has limits such as $y=a^n$ and $y=b^n$ and y-axis. The ratio of the volumes formed by the function and these boundaries will be equal to* ***n: 2.***

Also, I can show some examples, using *Advanced Calculator Software*, basically same functions, but rotating around y -axis now. That's what I got:

Function	V_A	V_B	Ratio
$X= y^{1/3}$	4.09	24.79	1 : 6
$X=y^{1/6}$	3.60	43.20	1 : 12
$X=y^{-1/2}$	2.18	8.72	1 : 4
$X=y^{-1/5}$	2.70	27	1 : 10
$X=y^2$	19.48	19.49	1 : 1
$X=y^{1.5}$	11.78	15.71	3 : 4

Tab 4This table, also proves that conjecture works, I decided to cal culate that, just in order to make sure. It's kind of a method for proving the conjecture, by collecting the relevant data.

Conclusion

In this investigation, I was considering power functions of the type y=x ^{n}, where n$\in$R. I had to find, ratios of the are as and volumes, with given boundaries. My limitations are basically, that the boundaries are positive, so *a and b >0.* The conjectures were proved, by using the method of math induction. I got the following results:

$\therefore$ If the power function of the type y=x^{n}, where n – is the power,

- *where n – is the power, n $\in$Z^{+} and has limits such as x=0 and x=1 and x -axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to* ***n: 1.***
- *where n – is the power, n $\in$Z (where $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ and has limits such as x=0 and x=1 and x-axis; y=0 and y=1 and the y-axis. The ratios of the areas formed by the function and by the boundaries will be equal to* ***1: |n|.***
- *where n – is the power, n $\in$Q (where Q=$\left\{ x \middle| x = \frac{a}{b}, b \neq 0\right.$ and a and b are integers} and has limits such as x=0 and x=1 and x -axis; y=0 and y=1 and the y -axis. The ratios of the areas formed by the function and by the boundaries will be equal to* ***1:***$\frac{1}{|n|}$

$\therefore$ If the power functions of the type y=x^{n}, where n – is the power, n$\in$R and has limits such as x=a and x=b and x-axis. The ratio of the volumes (rotated about x-axis) formed by the function and these boundaries will be equal to ***2 n: 1.***

$\therefore$ If the power functions of the type y=x^{n}, where n – is the power, n$\in$R and has limits such as y=a^{n} and y=b^{n} and y-axis. The ratio of the volumes (rotated about y -axis) formed by the function and these boundaries will be equal to ***n: 2***