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Math Portfolio SL Type 1:

Matrix Binomials

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This portfolio deals with the investigation of matrix binomials. This portfolio will cover the basic knowledge of matrix in order to explain the methods and the procedure of arriving at the general statement. The task was to deduce a general formula for given matrices and to determine scope and/or limitation of this general statement. The general statement was obtained by using an algebraic method. All matrix equations were typed using the program MathType. It will test the validity of the general statement using GDC (GDC-TI-83). The scope and limitations of this general statement will also be discussed in this investigation.

Matrix is "a rectangular array of numbers arranged into a fixed number of rows (horizontal) and columns (vertical)." Each number in the array or the collection of numbers is called an element or entry of the matrix. The order of elements of the matrix is: $\begin{pmatrix} \#1 & \#2 \\ \#3 & \#4 \end{pmatrix}$.

Matrix has dimension of m (rows) x n (columns). The numbers m and n are the dimensions of the matrix and it plays significant role in the calculation of the matrix. For instance, the dimension of the following matrix is (2X2) since it has two rows and two columns: $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

For the addition of the matrices, corresponding elements need to be added. For example, If A and B were two different matrices but with the same dimensions, each elements are added according to its position of the matrix or the order of the elements. To add these two matrices, elements according to its order or position will be added; however, it needs to be remembered that this will only work when both matrices have same dimensions.

Example, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

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Scalar (real number) multiplication is the multiplication of the matrix using real number. In this case, a real number is multiplied by each element in the matrix one by one.

Example: $3\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$$3\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3\times 2 & 3\times 3 \\ 3\times 1 & 3\times 4 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 9 \\ 3 & 12 \end{pmatrix}$$

The multiplication of matrices is very different from the addition or the scalar multiplication of matrices. In the case of addition and the scalar multiplication, elements in the matrices had to be either added or multiplied according to their corresponding elements; however, this rule does not apply for the multiplication of matrices. It is based on row-by-column multiplication. For the multiplication of matrices, multiply the first element in the row by first element in the column. Then multiply the second element in the row by the second element in the column. This continues and in the last stage, products are added; however, this method only works when rows and columns have the same number of elements.

Example: $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1\times 3 + 2\times 4 \end{pmatrix}$$
$$= \begin{pmatrix} 3+8 \end{pmatrix}$$
$$= \begin{pmatrix} 11 \end{pmatrix}$$

So, when both matrices A and B are multiplied, their rows and columns should have the same number of elements. For instance, the number of elements in the row of the matrix A should be equal to the number of elements in the column of the matrix B. Knowing the dimensions of matrices is useful for the multiplication of matrices.

$$A_{m\times n} \times B_{n\times p} = (AB)_{m\times p}$$

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Numbers in ‘n’ should be equal and numbers in ‘m’ and ‘p’ give the dimension of AB.

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$A_{(3\times2)} \times B_{(2\times2)}$

Since numbers in each column and in each row of matrices are the same, the dimension of AB will be $A \times B_{(3\times2)}$.

$$
\begin{aligned}
A \times B_{(3\times2)} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \\
&= \begin{pmatrix} (1\times2)+(2\times2) & (1\times2)+(2\times2) \\ (3\times2)+(4\times2) & (3\times2)+(4\times2) \\ (5\times2)+(6\times2) & (5\times2)+(6\times2) \end{pmatrix} \\
&= \begin{pmatrix} 2+4 & 2+4 \\ 6+8 & 6+8 \\ 10+12 & 10+12 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 6 \\ 14 & 14 \\ 22 & 22 \end{pmatrix}
\end{aligned}
$$

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Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

By calculating powers of X and Y, a definite relationship was found between the power of the matrix and the elements. When X^2, all the elements in the matrix were 2; all the elements were 4 when X^3; all the elements were 8 when X^4. There was a similar pattern in Y matrix as well. Everything was similar but in this case, there were two negative signs; however, these two negative signs did not change their position but remained in the same position throughout.

$$\begin{aligned} X^2 &= X \times X \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Y^2 &= Y \times Y \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} X^3 &= X^2 \times X \\ &= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Y^3 &= Y^2 \times Y \\ &= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} X^4 &= X^3 \times X \\ &= \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Y^4 &= Y^3 \times Y \\ &= \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} \end{aligned}$$

The following expressions, X^n and Y^n, were found by finding patterns when X and Y were powered. By examining the previous result, the elements in the matrix doubled as the number of power increased. It was also found that the end products of the matrix were the result of two to the power of one less than the power of X.

$$X^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

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To test this expression, ‘n’ was replaced with 3. Previously, when X was to the power of 3, all elements were 4.

Example:

$$X^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

If $n = 3$,

$$X^3 = \begin{pmatrix} 2^{3-1} & 2^{3-1} \\ 2^{3-1} & 2^{3-1} \end{pmatrix}$$

$$= \begin{pmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

This expression was tested by replacing ‘n’ with 3. The result obtained from using the expression and the result obtained from previous section all had elements of 4. Therefore, the expression that was formed is correct.

In the case of Y matrix, it was little different due to the negative signs. The Y matrix had similar patterns as the X matrix except for the two negative signs in the second and the third element. Therefore, for Y^n, negative signs should be add in the position of the second and the third element.

$$Y^n = \begin{pmatrix} 2^{n-1} & (-)2^{n-1} \\ (-)2^{n-1} & 2^{n-1} \end{pmatrix}$$

To test this expression, ‘n’ was replaced with 3. Previously, when Y was to the power of 3, the first and the fourth elements were 4 and the second and the third were -4.

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Example:

$$Y^n = \begin{pmatrix} 2^{n-1} & (-)2^{n-1} \\ (-)2^{n-1} & 2^{n-1} \end{pmatrix}$$

If $n = 3$,

$$Y^3 = \begin{pmatrix} 2^{3-1} & (-)2^{3-1} \\ (-)2^{3-1} & 2^{3-1} \end{pmatrix}$$

$$= \begin{pmatrix} 2^2 & (-)2^2 \\ (-)2^2 & 2^2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$$

This expression was tested by replacing 'n' with 3. The result obtained from using the expression and the result obtained from previous section were the same. They all had elements of 4 in the first and the last order of the element while they contained -4 in the second and the third order of the elemnt. Therefore, the expression that was formed is correct.

For $(X+Y)^n$, the actual values of X and Y were added in order to see the pattern. The result obtained after the calculation was an "identity matrix," which is a scalar matrix in which all of the diagonal elements are unit. It is usually represented by 'I.' When a matrix is multiplied by an identity matrix, the matrix remains unchanged. This is the property of an identity matrix.

Example: $A = \begin{pmatrix} 2 & 3 \\ -9 & 4 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A \times I = \begin{pmatrix} 2 & 3 \\ -9 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ -9 & 4 \end{pmatrix}$$

This makes it easier to make an expression for $(X+Y)^n$ due to this property of identity matrix.

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$$(X+Y)^n = \left(\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}+\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\right)^n$$
$$= \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}^n$$
$$= \left(2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)^n$$
$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$
$$2^n\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2^n & 0\\0 & 2^n\end{pmatrix}$$

Therefore,

$$(X+Y)^n = \begin{pmatrix}2^n & 0\\0 & 2^n\end{pmatrix}$$

Let A=aX and B=bY, where a and b are constants.

There was a pattern found when A and B were each powered. For example, when A was to the power of 2, the result showed that both constant and the matrix were powered by 2 as well. When X^3, both constant and the matrix were powered by 3. This same pattern was applied to B as well.

$$A^2 = (aX)^2$$
$$= a^2X^2$$
$$A^3 = (aX)^3$$
$$= a^3X^3$$

$$A^4 = (aX)^4$$
$$= a^4X^4$$

$$B^2 = (bY)^2$$
$$= b^2Y^2$$
$$B^3 = (bY)^3$$
$$= b^3Y^3$$

$$B^4 = (bY)^4$$
$$= b^4Y^4$$

Similarly, the pattern was found in order to find the expression for A^n, B^n and $(A+B)^n$. It needs to

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be remembered that X and Y are not constants but matrix. So when both A and B were to the power of 'n', both constants and matrix were powered by n too. From the previous section, the expression for both X^n and Y^n were found. These expressions for X^n and Y^n were replaced by general expression:

$$\begin{aligned} A^n &= a^n X^n \\ &= a^n \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} B^n &= b^n Y^n \\ &= b^n \begin{pmatrix} 2^{n-1} & (-)2^{n-1} \\ (-)2^{n-1} & 2^{n-1} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (A+B)^n &= (aX+bY)^n \\ &= \left(\left(\begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) + \left(\begin{pmatrix} b & b \\ b & b \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)\right)^n \\ &= \left(\begin{pmatrix} a & a \\ a & a \end{pmatrix} + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}\right)^n \\ &= \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n \end{aligned}$$

From this, it can be found that (A+B) is equal to the matrix M since they have the same matrix.

$$M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

$$\text{(A+B)} = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

To show that $M^2=A^2+B^2$, the matrix M was squared. After that the matrix A and B were both squared in order to compare with the matrix M.

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$$M^2 = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \times \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$
$$= \begin{pmatrix} (a+b)^2 + (a-b)^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & (a+b)^2 + (a-b)^2 \end{pmatrix}$$

Since: $(a+b)^2 + (a-b)^2 = a^2 + b^2 + 2ab + a^2 + b^2 - 2ab$

$$= \begin{pmatrix} 2(a^2+b^2) & 2(a^2+b^2) \\ 2(a^2-b^2) & 2(a^2+b^2) \end{pmatrix}$$
$$= 2\begin{pmatrix} (a^2+b^2) & (a^2+b^2) \\ (a^2-b^2) & (a^2+b^2) \end{pmatrix}$$

$$A^2 = a^2(X^2)$$
$$= \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}$$

$$B^2 = b^2(Y^2)$$
$$= \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}$$

For A^2 and B^2, the matrix X and Y were replaced with the values given in the first section of investigation.

$$A^2 + B^2 = \begin{pmatrix} a & a \\ a & a \end{pmatrix}^2 + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}^2$$
$$= \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix} + \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}$$
$$= \begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$$
$$= 2\begin{pmatrix} (a^2+b^2) & (a^2+b^2) \\ (a^2-b^2) & (a^2+b^2) \end{pmatrix}$$

By expanding the expression, the process of $M^2=A^2+B^2$. is shown.

To get the expression of M^n in terms of aX and bY, the matrix M was squared and cubed. This

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was done to see the patterns, which are significant in finding the expression. When there was the part for the algebraic method, Pascal's Triangle was used in order to ease the expansion of the equation.

$$M^n = ?$$

$$M^2 = 2\begin{pmatrix} a^2+b^2 & a^2-b^2 \\ a^2-b^2 & a^2+b^2 \end{pmatrix}$$

$$M^3 = M^2 \times M$$

$$= 2\begin{pmatrix} a^2+b^2 & a^2-b^2 \\ a^2-b^2 & a^2+b^2 \end{pmatrix} \times \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

$$= 2\begin{pmatrix} (a^2+b^2)(a+b)+(a^2-b^2)(a-b) & (a^2+b^2)(a-b)+(a^2+b^2)(a-b) \\ (a^2+b^2)(a-b)+(a^2+b^2)(a-b) & (a^2+b^2)(a+b)+(a^2-b^2)(a-b) \end{pmatrix}$$

$$(a^2+b^2)(a+b)+(a^2-b^2)(a-b) = a^3+ab^2+a^4b+b^3+a^3-ab^2-a^2b+b^3 = 2a^3+2b^3$$

$$= 2\begin{pmatrix} 2a^3+2b^3 & 2a^3-2b^3 \\ 2a^3-2b^3 & 2a^3+2b^3 \end{pmatrix}$$

$$= 2\times 2\begin{pmatrix} a^3+b^3 & a^3-b^3 \\ a^3-b^3 & a^3+b^3 \end{pmatrix}$$

$$M^n = 2^{n-1}\begin{pmatrix} a^nX+b^nY & a^nX-b^nY \\ a^nX-b^nY & a^nX+b^nY \end{pmatrix}$$

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To test this expression, ‘a’ was replaced with 2, ‘b’ was replaced with 4 and ‘n’ was replaced with 3.

$$M^n = 2^{n-1}\begin{pmatrix} a^nX + b^nY & a^nX - b^nY \\ a^nX - b^nY & a^nX + b^nY \end{pmatrix}$$

$$M^3 = 2^{3-1}\begin{pmatrix} (2)^3 + (4)^3 & (2)^3 - (4)^3 \\ (2)^3 - (4)^3 & (2)^3 + (4)^3 \end{pmatrix}$$

$$= 2^2\begin{pmatrix} 8+64 & 8-64 \\ 8-64 & 8+64 \end{pmatrix}$$

$$= 4\begin{pmatrix} 72 & -56 \\ -56 & 72 \end{pmatrix}$$

To test it, ‘a’ was replaced with 2, ‘b’ was replaced with 4 for the M^3 in the previous section in order to compare the result.

$$M^3 = 2\times 2\begin{pmatrix} a^3 + b^3 & a^3 - b^3 \\ a^3 - b^3 & a^3 + b^3 \end{pmatrix}$$

$$= 4\begin{pmatrix} (2)^3 + (4)^3 & (2)^3 - (4)^3 \\ (2)^3 - (4)^3 & (2)^3 + (4)^3 \end{pmatrix}$$

$$= 4\begin{pmatrix} 8+64 & 8-64 \\ 8-64 & 8+64 \end{pmatrix}$$

$$= 4\begin{pmatrix} 72 & -56 \\ -56 & 72 \end{pmatrix}$$

Since the result was the same, the expression that was formed was true.

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The significance of the matrix is that column vectors and row vectors become the basis for the vector space. This is very important when dealing with dimensions bigger than $n \geq 2$ (for 3 dimension). In the case of mathematical application, the matrix is useful and significant in solving multivariable equation system such as $\begin{matrix} 3x+5y+4z=14 \\ 8x+2y+2z=32 \end{matrix}$.

Matrices can also be used with encryption. Programmers often use matrices and their inverse to either encrypt or code messages. Many geologists also made specific types of matrices for seismic surveys. Matrices are also used in banks to transmit sensitive or private data. In economy, matrices are used to calculate GDP (Gross Domestic Products) to calculate produced goods more efficiently. For scientific studies and for various fields, matrices are frequently used in statistics and graphs. Matrices are sometimes used in computer animation.

There were some scope and the limitation that could be found in this general statement of M^n. It is that 'n' can't be less than 0 and must be natural numbers because this matrix can't have a negative value and will not work if it's complex number or irrational number. The value of a and b should be natural numbers because the matrix will not work if it's complex number or irrational number. Additionally, 'a' cannot equal to both 'b' and 'n.' If this happens, the general formula for M^n will not work. A limitation was found when using the GDC. Although GDC was useful in calculating numbers, it could not find a pattern and form a general formula. Additionally, it could not use algebraic coefficients like 'a,' 'b,' and 'n' to find the general statement.

To form this general statement, the Pascal's Triangle was used to expand the algebraic equations. Pascal's triangle is 'a geometric arrangement of the binomial coefficients in a triangle.' Since it is very complicated to multiply all those algebraic form of the equation, this triangle was used to ease the expansion of the algebraic form. The calculator was used for the simple mathematical calculations and to test the general statement. The program MathType was used to type in all forms of the matrices in the computer.

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