Infinite Surds

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Math SL Portfolio
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$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$$

Previous expression is known as an infinite surd. We can turn the surd in to a following sequence:

$$a_1 = \sqrt{1+\sqrt{1}} = 1,414213$$

$$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}} = 1,553773$$

$$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} = 1,598053$$

$$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}} = 1,611847$$

$$a_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}} = 1,616121$$

$$a_6 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}} = 1,617442$$

$$a_7 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}} = 1,617851$$

$$a_8 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}} = 1,617977$$

$$a_9 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}} = 1,618016$$

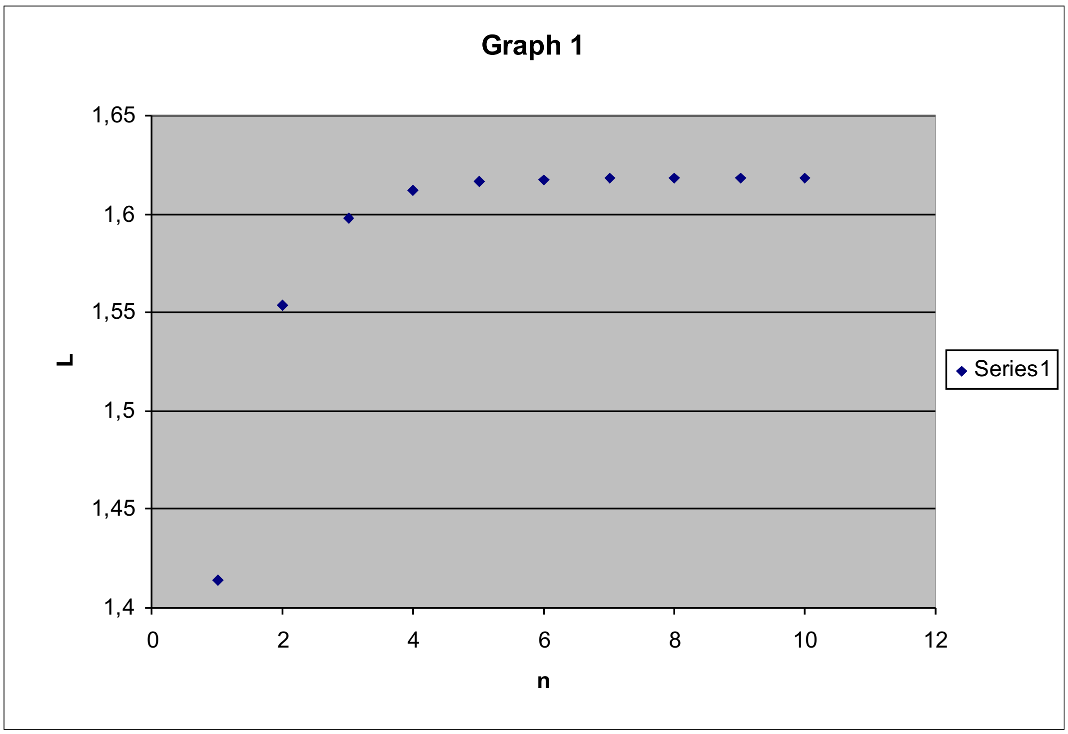
$$a_{10} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}}} = 1,618028$$

etc.

From the 10 first terms of the sequence we can present the relation of two following terms as:

$a_{n+1} = \sqrt{1 + a_n}$

If we would like to study the relation between $a_n - a_{n+1}$, plotting a graph from the 10 first terms of the sequence might bring some light to the matter. Graph 1 illustrates the relation between n and L in the case that $L = a_n$



As we can see from the graph the value of L slowly moves toward value of approximately 1,618 or so, but will never actually reach it. If we furthermore consider what this shows about the relation between a_n and a_{n+1}, we can determine that in the case of

$a_n - a_{n+1}$

when n approaches infinity

lim($a_n - a_{n+1}$) → 0

We can expand our point by taking another sequence build from infinite surd as an example. Let our sequence be following.

$$b_1 = \sqrt{2+\sqrt{2}} = 1,847759065$$

$$b_2 = \sqrt{2+\sqrt{2+\sqrt{2}}} = 1,961570561$$

$$b_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = 1,990369453$$

$$b_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} = 1,997590912$$

$$b_5 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}} = 1,999397637$$

$$b_6 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} = 1,9998494404$$

$$b_7 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}} = 1,9999849404$$

$$b_8 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}} = 1,999990588$$

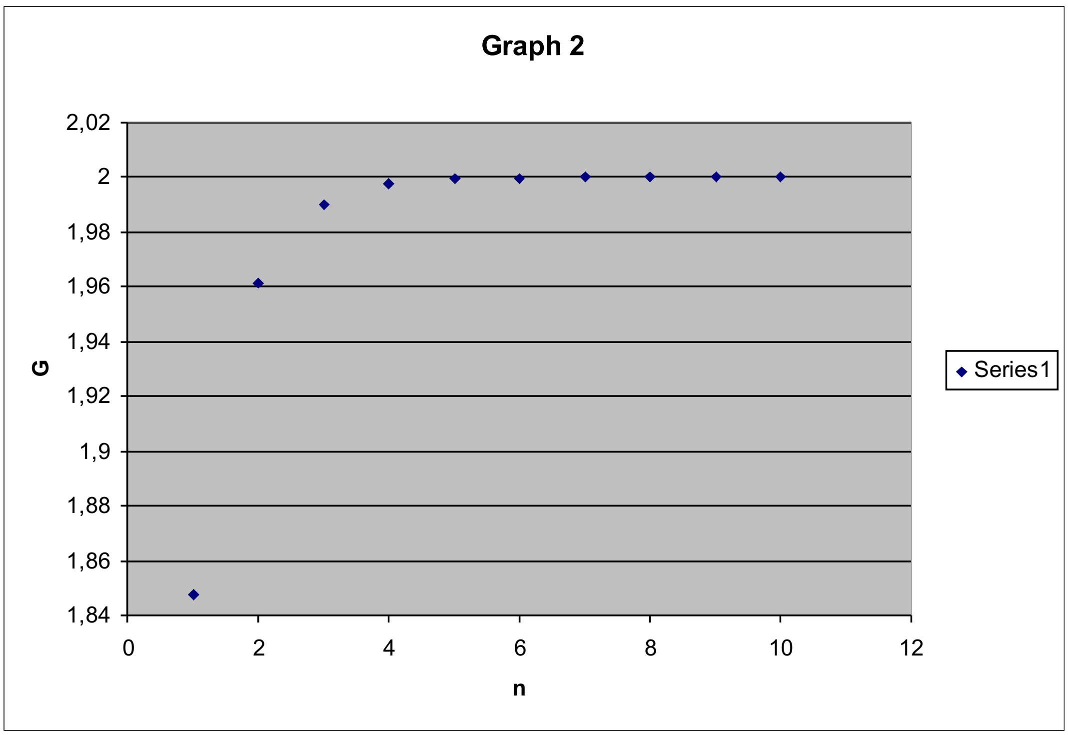
$$b_9 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}} = 1,999997647$$

$$b_{10} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}} = 1,999999412$$

etc.

Graph 2 illustrates G in the case that

$G = a_n$



From here we can easily see that a_n approaches value of 2 but never quite reaches it. Let us now consider a general example:

$$x = \sqrt{k+\sqrt{k+\sqrt{k..}}}$$
$$x^2 = \sqrt{k+\sqrt{k+\sqrt{k..}}}^2$$
$$x^2 = k+\sqrt{k+\sqrt{k+\sqrt{k..}}}$$

Because of the fact that we are working with an infinite surd:

$x^2 = k + x$

$0 = k + x - x^2$

$0 = (x + k)(x - k)$

And from here we can deduce by using the null factor law the expression for any value of k which expression forms an integer. Examples:

$(x + 2)(x - 2) = 0$

Because we know that the expression is positive we can determine that

$x - 2 = 0$

$x = 2$

When we compare this result to our results present in the graph to we can determine that our general term works.