Investigating Divisibility

In order to determine if an expression is divisible by a certain value, we factorize the expression and see if we can take the corresponding value let's call it x as a common factor. Afterwards, we see if it is divisible depending on how the expression will turn out. I'll explain more with examples.

Now let's look at the expression $P(n) = n^x - n \quad x \in \{2,3,4,5\}$. Now we want to see if the expression is always divisible by the corresponding x .

The first case if x =2.

Now by substituting 2 in the expression, the expression will look like this:

$P(n) = n^2 - n$

Now let's take n as a common factor. The expression will become $n(n-1)$ and since the expression $P(n) = odd \ * even$ or $P(n) = even \ * odd$ it is therefore divisible by 2. Now let's check the validity of my statement let's take a few examples.

Let $n = 5, 14, 20, 51$

Using GDC we plug the following values in the expression and check if it is divisible by 2.

$P(5) = 5^2 - 5 = 20$ And 20 is divisible by 2.
$P(14) = 14^2 - 14 = 182$ Again 182 is divisible by 2.
$P(20) = 20^2 - 20 = 380$ Also 280 is divisible by 2
$P(51) = 51^2 - 51 = 2550$ Which is divisible by 2

Therefore $P(n)$ is divisible by 2.

Now let's take the second case when $x = 3$

Now by substituting 3 in the expression it will turn out to be like this:

$P(n) = n^3 - n$

Let's take n as a common factor the expression will now look like this:

$P(n) = n(n^2 - 1)$ And now by factorizing it more where $(n^2 - 1)$ is difference between two squares, the expression will look like this $P(n) = n(n-1)(n+1)$ which is three successive (consecutive) terms.

Therefore, $P(n)$ is divisible by 3.

To make sure this is true let's take a few examples.

Let $n = 37, 98, 111$

Now using GDC substitute the following values of n in the expression

$P(37) = 37(37-1)(37+1) = 50616$ Which is divisible by 3
$P(98) = 98(98-1)(98+1) = 941094$ And this number is divisible by 3
$P(111) = 111(111-1)(111+1) = 1367520$ Again this number is divisible by 3.

Therefore $P(n)$ is divisible by 3

The 3rd case is when $x = 4$

Now by plugging 4 in the expression it will turn out to be like this:

$P(n) = n^4 - n$

And now by factorizing it, the expression will look like this: $P(n) = n(n^3 - 1)$

Now by factorizing the expression further it will look like this:

$P(n) = n(n-1)(n^2 + n + 1)$

I can't find anything in the expression that shows that it is divisible by 4. However,

Let's take some examples to check.

Let $n = 2,7$

By using GDC will plug the following values of n in the expression and check if it is divisible by 4.

$P(2) = 2(1)(2^2 + 2 + 1) = 14$ Which is not divisible by 4
$P(7) = 7(7-1)(7^2 + 7 + 1) = 294$ And this number is not divisible by 4.

Therefore $P(n)$ is not divisible by 4.

Let's look at the fourth case which is $x = 5$

Now let's plug the value of x in the expression.

$P(n) = n^5 - n$ Now let's keep factorizing the expression more

$P(n) = n(n^4 - 1)$

$P(n) = n(n^2 - 1)(n^2 + 1)$

$P(n) = n(n - 1)(n + 1)(n^2 + 1)$

I don't find any clear evidence in the expression to show if the expression is divisible by 5 or not however, let's take a few example to check.

Let $n = 5, 14$

Now by plugging the following n values in the expression we check if it is divisible by 5.

$P(5) = 5(5 - 1)(5 + 1)(5^2 + 1) = 3120$ which is divisible by 5

$P(14) = 14(14 - 1)(14 + 1)(14^2 + 1) = 57810$ Which is divisible by 5.

Therefore $P(n)$ is divisible by 5.

Using mathematical induction I'm going to prove whether $P(k+1)-P(k)$ is always divisible by the x that $P(n)$ was divisible by. The values of x that $P(n)$ was divisible by were 2, 3 and 5. Therefore using mathematical induction I'm going to check if $P(k+1)-P(k)$ is always divisible for these values.

The First case is $x=2$

Now by using mathematical induction we want to prove that $P(k+1)-P(k)$ is divisible by 2.

First we let n=1 and see if it is divisible by 2

$P(n)=1^2-1=0$ Therefore divisible by 2,

Now we assume that $n=k$ is true so $P(k)$ is true.

However, we need to prove that $P(k+1)-P(k)$ is true.
$P(k+1)=(k+1)^2-(k+1)$ And $P(k)=(k)^2-k$.

Now let's subtract $P(k)$ from $P(k+1)$.

The expression is now $(k+1)^2-(k+1)-[(k)^2-k]$

Let's solve the brackets, the expression becomes in this form:

$k^2+2k+1-k-1-k^2+k$ By collecting terms and simplifying the

Expression it will become:

$2k$

Which is indeed divisible by 2

The second case is $x = 3$

Now by using mathematical induction we want to prove that $P(k+1) - P(k)$ is divisible by 3.

First we let n=1 and see if it is divisible by 3

$P(n) = 1^3 - 1 = 0$ Therefore divisible by 3.

Now we assume that $n = k$ is true so $P(k)$ is true.

However, we need to prove that $P(k+1) - P(k)$ is true.
$P(k+1) = (k+1)^3 - (k+1)$ And $P(k) = (k)^3 - k$.

Now let's subtract $P(k)$ from $P(k+1)$.

The expression is now $(k+1)^3 - (k+1) - [(k)^3 - k]$

Let's solve the brackets so the expression becomes in this form:

$k^3 + 3k^2 + 3k + 1 - k - 1 - k^3 + k$ By collecting terms and simplifying the

Expression it will become:

$3k^2 + 3k$

And by taking $3k$ as common factor the expression is now:

$3k(k+1)$

Therefore it is divisible by 3

Since $P(n)$ is not divisible by 4 it is ignored and we don't have to prove by induction.

The 3rd case is when $x = 5$

Now by using mathematical induction we want to prove that $P(k+1) - P(k)$ is divisible by 5.

First we let n=1 and see if it is divisible by 5

$P(n) = 1^5 - 1 = 0$ Therefore divisible by 5.

Now we assume that $n = k$ is true so $P(k)$ is true.

However, we need to prove that $P(k+1) - P(k)$ is true.
$P(k+1) = (k+1)^5 - (k+1)$ And $P(k) = (k)^5 - k$.

Now let's subtract $P(k)$ from $P(k+1)$.

The expression is now $(k+1)^5 - (k+1) - [(k)^5 - k]$

Let's solve the brackets, so the expression becomes in this form:

$k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 - k^5 + k$ By collecting terms and simplifying the

Expression it will become:
$5k^4 + 10k^3 + 10k^2 + 5k$

And by taking $5k$ as common factor the expression is now:

$5k(k^3 + 2k^2 + 2k + 1)$

Therefore it is divisible by 5.

Now let's explore more cases for x and if $P(n)$ is divisible by x we'll prove it by induction. So we will factorize the expression $P(n) = n^x - n$ for $x \in \{6,7,11\}$

Let's look at when $x = 6$

Now let's plug the value of x in the expression.

$P(n) = n^6 - n$ Now let's keep factorizing the expression more
$P(n) = n(n^5 - 1)$

I don't find any clear evidence in the expression to show if the expression is divisible by 6 or not however, let's take a few example to check.

Let $n = 2,9$

Now by plugging the following n values in the expression we check if it is divisible by 6.

$P(2) = 2(2^5 - 1) = 62$ which is not divisible by 6
$P(9) = 9(9^5 - 1) = 531432$ Which is not divisible by 6.

Therefore $P(n)$ is not divisible by 6 which is not prime.

Let's look at when $x = 7$
Now let's plug the value of x in the expression.
$P(n) = n^7 - n$ Now let's keep factorizing the expression more
$P(n) = n(n^6 - 1)$

I don't find any clear evidence in the expression to show if the expression is divisible by 7 or not however, let's take a few example to check.

Let $n = 10, 15$

Now by plugging the following n values in the expression we check if it is divisible by 7.

$P(10) = 10(10^6 - 1) = 999990$ which is divisible by 7
$P(9) = 15(15^6 - 1) = 1708560$ Which is divisible by 7.

Therefore $P(n)$ is divisible by 7.

Now by using mathematical induction we want to prove that $P(k+1) - P(k)$ is divisible by 7.

First we let n=1 and see if it is divisible by 7

$P(n) = 1^7 - 1 = 0$ Therefore divisible by 7.

Now we assume that $n = k$ is true so $P(k)$ is true.

However, we need to prove that $P(k+1) - P(k)$ is true.
$P(k+1) = (k+1)^7 - (k+1)$ And $P(k) = (k)^7 - k$.

Now let's subtract $P(k)$ from $P(k+1)$.

The expression is now $(k+1)^7 - (k+1) - [(k)^7 - k]$

Let's solve the brackets so the expression becomes in this form:

$k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1 - k^7 + k$ By collecting terms and simplifying the expression it will become:

$7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k$

And by taking $7k$ as common factor the expression is now:

$7k(k^5 + 3k^4 + 5k^3 + 5k^2 + 3k + 1)$

Therefore it is divisible by 7.

Let's look at when $x = 11$
Now let's plug the value of x in the expression.
$P(n) = n^{11} - n$ Now let's keep factorizing the expression more
$P(n) = n(n^{10} - 1)$

I don't find any clear evidence in the expression to show if the expression is divisible by 11 or not however, let's take a few example to check.

Let $n = 4,5$

Now by plugging the following n values in the expression we check if it is divisible by 11.

$P(4) = 4(4^{10} - 1) = 4194300$ Which is divisible by 11
$P(9) = 5(5^{10} - 1) = 48828120$ Which is divisible by 11.

Therefore $P(n)$ is divisible by 11.

Now by using mathematical induction we want to prove that $P(k+1) - P(k)$ is divisible by 11.

First we let n=1 and see if it is divisible by 11

$P(n) = 1^{11} - 1 = 0$ Therefore divisible by 11.

Now we assume that $n = k$ is true so $P(k)$ is true.

However, we need to prove that $P(k+1) - P(k)$ is true.
$P(k+1) = (k+1)^{11} - (k+1)$ And $P(k) = (k)^{11} - k$.

Now let's subtract $P(k)$ from $P(k+1)$.

The expression is now $(k+1)^{11} - (k+1) - [(k)^{11} - k]$

Let's solve the brackets so the expression becomes in this form:

$k^{11} + 11k^{10} + 55k^9 + 165k^8 + 330k^7 + 462k^6 + 264k^5 + 330k^4 + 165k^3 + 55k^2 + 11k + 1 - k - 1 - k^{11} + k$
By collecting terms and simplifying the expression it will become:

$11k^{10} + 55k^9 + 165k^8 + 330k^7 + 462k^6 + 264k^5 + 330k^4 + 165k^3 + 55k^2 + 11k$

And by taking $11k$ as common factor the expression is now:

$11k(k^9 + 5k^8 + 15k^7 + 30k^6 + 42k^5 + 24k^4 + 30k^3 + 15k^2 + 5k)$

Therefore it is divisible by 11.

After looking at several cases I came up with a conjecture that if x in the expression $P(n) = n^x - n$ is a prime number then $P(n)$ is divisible by x.

Now let's look at Pascal's triangle to find a pattern and explain how the entries are obtained.

Pascal's rules states that using the binomial coefficient which is $\binom{x}{r} = \frac{(x)!}{(r)!\,(x-r)!}$ where x is the number of rows and r is the position in the row. Starting with x=0. For example in order to obtain the second entry in the 2nd row we plug the following in this equation $\binom{2}{2}\frac{(2)!}{(2)!\,(3-2)!}$ which is equal to 2. In the Pascal triangle each row starts and ends with 1 and there is infinite number or rows. And the entries increases by 1 as you increase the number of rows. For example, the first row has 0 entries and the second row has 1 entries and the third has 2 entries and the forth has 3 entries and so on. So if you want to generate the 3rd row for example you know that it has 2 entries and it start with 1 and ends with 1 so you need to find the first and 2nd entry. You can find them using Pascal's rule. The first entry is 3 using the Pascal's rule where you plug the number of row and the entry's number in the equation. So the first entry in the 3rd row is $\binom{3}{1}\frac{(3)!}{(1)!\,(3-1)!}$ which is equal to 3 and the second entry is found by plugging the number of rows and the position in the row in the equation and it will turn out to be: $\binom{3}{2}\frac{(3)!}{(2)!\,(3-2)!}$ which is also equal to 3. Therefore the first entry will be 3 and the second entry is 3. Thus, the row will look like this 1 3 3 1.

Using GDC I can apply Pascal's rule using the $_nC_r$ where n is the number of rows and r is the position in the row. And so I can generate the first 15 rows using GDC. For example, let's say that I want to generate the 3rd row, I know that it has 2 entries and that it starts with 1 and ends with 1.Therefore in order to find the first entry I click on 3 on the calculator as it is the number of rows and then I click on the math button and go to the PRB tab scrolling down to the third option $_nC_r$ and then I click the on the number of entry I want to find. In this case 1 and so it will look like this on the calculator screen $3\,_nC_r$ 1. Afterwards I click enter and it will give me 3 which is the first entry in the 3rd row. I will repeat this process for the first 15 rows and I can find them using this technology.

Here are the first 15 rows of the Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

I found a relationship between Pascal's triangle and the expression $P(k+1)-P(k)$. The coefficients of k in the expression $P(k+1)-P(k)$ are the entries in the x row of Pascal's triangle and x is the number of rows. For example if you want to find the entries of the 5th row you plug 5 in the expression $P(k+1)-P(k)$ instead of x so the expression is now $(k+1)^5-(k+1)-[(k)^5-k]$ by simplifying the expression it will look like this: $5k^4+10k^3+10k^2+5k$ and 5 10 10 5 are the entries in the fifth row and since each row starts and ends with 1 you can find the row and its entries by this relation. So the fifth row of Pascal's triangle will look like this 1 5 10 10 5 1. In addition, there's another relationship between Pascal's triangle and $P(k+1)-P(k)$, if x the number of rows is a prime number the entries in the row will be divisible by x. For example, let's look at the 3rd row of Pascal's triangle. The entries in the 3rd row 1 3 3 1 are divisible by 3 since 3 is a prime number. The same is with the expression $P(k+1)-P(k)$ if x is a prime number then the expression $P(k+1)-P(k)$ is divisible by x. Therefore, from this relationship we can conclude that k is a multiple of x if x is a prime number.

After finding this relationship I came up with a new conjecture. If x in the expression $P(n)=n^x-n$ is a prime number then $P(n)$ is divisible by x. Also, if x is a prime number in Pascal's triangle where x is the number of rows then the entries in the row are divisible by x and it is a multiple of k.

However, if we look at the converse of this statement which is if expression $P(n) = n^x - n$ divides by x then x is a prime number. This statement is not true since this expression can divide by some non-prime numbers for some values of n.

For example let's take 4, although 4 is not prime yet if you plug 5 in the expression $P(n) = n^x - n$ instead of n it will divide by 4.

$P(5) = 5^4 - 5 = 620$ which is divisible by 4.

Therefore, the converse of my conjecture is not true and doesn't hold.