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IB Math Standard Level Portfolio Assignment
Type 1- Mathematical Investigation
Logarithm Bases

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I have completed this assignment in accordance with the Newark Academy honor code.
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This investigation will determine the relationship between different sets of sequences including logarithms. This investigation will be tested using technology and general mathematic relationships.

There are a few rules, which govern all the concepts of logarithms:

$\log_a b = c$, $a^c = b$ where a>0, a≠1, b>0

$$\log_a b = \frac{\log_c b}{\log_c a}$$

The sets of sequences are as follows:

log□8, log□8, log□8, log□□8, log□□8

log□81, log□81, log□□81, log□□81

log□25, log□□25, log□□□25, log□□□25

:
:
:

$\log_m m^k, \log_{m^2} m^k, \log_{m^3} m^k, \log_{m^4} m^k$

By following these sequences a pattern can be shown. The base of each term in the sequences changes, but the number in the logarithm remains constant. The following two terms of each sequence were determined:

log□8, log□8, log□8, log□□8, log□□8, log□□8, log□□□8

log□81, log□81, log□□81, log□□81, log□□□81, log□□□81

log□25, log□□25, log□□□25, log□□□25, $\log_{3125} 25$, log□□□□□25

:
:
:

$\log_m m^k, \log_{m^2} m^k, \log_{m^3} m^k, \log_{m^4} m^k, \log_{m^5} m^k, \log_{m^6} m^k$

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Find an expression for the n[th] term of each sequence. Write down your expression in the form , where p, q $\in \mathbf{Z}$. Justify your answers using technology.

Begin with the first sequence (log□8, log□8, log□8, log□□8, log□□8, log□□8, log□□□8) and determine an expression for the nth term:

1	2	3	4	5	6	7
2	2	2	2	2	2	2
2^1	2^2	2^3	2^4	2^5	2^6	2^7
2	4	8	16	32	64	128

The value 2 was used to determine the nth term.

1. $\log_{2^n} 8 = \log_{2^n} 2^3$

Apply the change of base formula that states:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Then apply the rule: $\log_c b^a = a \log_c b$

$$\frac{\log_2 2^3}{\log_2 2^n} = \frac{3\log_2 2}{n \log_n 2}$$

$\log_2 2$ cancels out on both sides.

What remains is $\frac{3}{n}$, where n represents the nth term in the sequence.

Here are two graphs to test validity:

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y= $\frac{3}{x}$

QuickTime™ and a decompressor are needed to see this picture.

y= $\frac{\log_2 2^3}{\log_2 2^n}$

QuickTime™ and a decompressor are needed to see this picture.

Both graphs are identical towards each other, which indicates that both functions of the graphs are the same.

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Next, consider the second sequence (log□81, log□81, log□□81, log□□81, log□□□81, log□□□81)

1	2	3	4	5	6	7
3	3	3	3	3	3	3
3^1	3^2	3^3	3^4	3^5	3^6	3^7
3	9	27	81	243	729	2187

2. $\log_{3^n} 81 = \log_{3^n} 3^4$

Use the change of base formula

$$\frac{\log_3 3^4}{\log_3 3^n}$$

Then apply the rule: $\log_c b^a = a\log_c b$

$$\frac{\log_3 3^4}{\log_3 3^n} = \frac{4\log_3 3}{n\log_3 3}$$

$\log_3 3$ cancels out on both sides.

What remains: $\frac{4}{n}$

Here are two graphs to test its validity and accuracy:

y= $\frac{4}{x}$

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y= $\frac{\log_3 3^4}{\log_3 3^n}$

Both graphs are identical towards each other, which indicates that both functions of the graph are the same.

Next, consider the third sequence (log□25, log□□25, log□□□25, log□□□25, log□□□25, $\log_{3125} 25$, log□□□□□25)

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1	2	3	4	5	6	7
5	5	5	5	5	5	5
5^1	5^2	5^3	5^4	5^5	5^6	5^7
5	25	125	625	3125	15625	78125

3. $\log_{5^n} 25 = \log_{5^n} 5^2$

Use the change of base formula:

$$\frac{\log_5 5^2}{\log_5 5^n}$$

Then apply the rule: $\log_c b^a = a \log_c b$

$$\frac{\log_5 5^2}{\log_5 5^n} = \frac{2\log_5 5}{n\log_5 5}$$

$\log_5 5$ cancels out on both sides

What remains: $\frac{2}{n}$

Here are two graphs to test its validity and accurateness:

Y= $\frac{2}{x}$

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decompressor
are needed to see this picture.

$$Y=\frac{\log_5 5^2}{\log_5 5^n}$$

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Both graphs are identical towards each other, which indicates that both functions of the graph are the same.

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4. Sequence expressed using the variables m, n, and k.

$\log_{m^n} m^k$

Use $\log_a b = \frac{\log_c b}{\log_c a}$ to change the base of the logarithm to ten.

$\frac{\log_m m^k}{\log_m m^n}$

$\log_m m$ cancels out.

What remains is $\frac{k}{n}$. Derived from this, it can be concluded that the general expression for the n^th term of each sequence in the form $\frac{p}{q}$ is $\frac{k}{n}$.

Justification of this statement using technology:

$\log_4 8 = \log_{2^2} 2^3$ $(\log_{m^n} m^k)$

$\frac{\log 8}{\log 4} = \frac{3}{2} = 1.5$

$\log_{27} 81 = \log_{3^3} 3^4$ $(\log_{m^n} m^k)$

$\frac{\log 81}{\log 27} = \frac{4}{3} = 1.33$

$\log_{\frac{1}{5}} 125 = \log_{5^{-1}} 5^3$ $(\log_{m^n} m^k)$

$\frac{\log 125}{\log \frac{1}{5}} = \frac{3}{-1} = -3$

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Now calculate the following, giving your answer in the form , where p, q, .

The answer was . This form will be used.

1. $\log_4 64 = \log_{2^2} 2^6$ $\log_{m^n} m^k$

$$\frac{\log 64}{\log 4} = \frac{6}{2}$$

$\log_8 64 = \log_{2^3} 2^6$ $\log_{m^n} m^k$

$$\frac{\log 64}{\log 8} = \frac{6}{3}$$

$\log_{32} 64 = \log_{2^5} 2^6$ $\log_{m^n} m^k$

$$\frac{\log 64}{\log 32} = \frac{6}{5}$$

2. $\log_7 49 = \log_{7^1} 7^2$ $\log_{m^n} m^k$

$$\frac{\log 49}{\log 7} = \frac{2}{1}$$

$\log_{49} 49 = \log_{7^2} 7^2$ $\log_{m^n} m^k$

$$\frac{\log 49}{\log 49} = \frac{2}{2}$$

$\log_{343} 49 = \log_{7^3} 7^2$ $\log_{m^n} m^k$

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$$\frac{\log 49}{\log 343} = \frac{2}{3}$$

3. $\log_{\frac{1}{5}} 125 = \log_{5^{-1}} 5^3$ $\log_{m^n} m^k$

$$\frac{\log 125}{\log \frac{1}{5}} = \frac{3}{-1}$$

$\log_{\frac{1}{125}} 125 = \log_{5^{-8}} 5^3$ $\log_{m^n} m^k$

$$\frac{\log 125}{\log \frac{1}{125}} = \frac{3}{-3}$$

$\log_{\frac{1}{625}} 125 = \log_{5^{-4}} 5^3$ $\log_{m^n} m^k$

$$\frac{\log 125}{\log \frac{1}{625}} = \frac{3}{-4}$$

4. $\log_8 512 = \log_{2^3} 2^9$ $\log_{m^n} m^k$

$$\frac{\log 512}{\log 8} = \frac{9}{3}$$

$\log_2 512 = \log_{2^1} 2^9$ $\log_{m^n} m^k$

$$\frac{\log 512}{\log 2} = \frac{9}{1}$$

$\log_{16} 512 = \log_{2^4} 2^9$ $\log_{m^n} m^k$

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$$\frac{\log 512}{\log 16} = \frac{9}{4}$$

Describe how to obtain the third answer in each row from the first two answers. Create two more examples that fit the pattern above.

1.

$$\log_4 64, \log_8 64, \log_{32} 64 = \log_{2^2} 2^6, \log_{2^3} 2^6, \log_{2^5} 2^6$$

By recognizing that 2^n is the base in each of these logarithm, it is apparent that n=1 in the first logarithm and n=2 in the second logarithm. When added 2+3=5, therefore the third answer is obtained by multiplying the bases together in accordance with the rules of exponents.

The next two examples that would fit the pattern would therefore be:

$$\log_{2^8} 2^6, \log_{2^{13}} 2^6 = \log_{256} 64, \log_{8192} 64$$

2.

$$\log_7 49, \log_{49} 49, \log_{343} 49 = \log_{7^1} 7^2, \log_{7^2} 7^2, \log_{7^3} 7^2$$

By recognizing 7^n is the base in each of these logarithms, it is apparent that n=1 is the first logarithm and n=2 in the second logarithm. When added 1+2=3, therefore the third answer is obtained by multiplying the bases together in accordance with the rules of exponents.

The next two examples that would fit the patter would therefore be:

$$\log_{7^4} 7^2, \log_{7^7} 7^2 = \log_{2401} 49, \log_{823543} 49$$

3.

$$\log_{\frac{1}{5}} 125, \log_{\frac{1}{125}} 125, \log_{\frac{1}{625}} 125 = \log_{5^{-1}} 5^3, \log_{5^{-3}} 5^3, \log_{5^{-4}} 5^3$$

By recognizing 5^n is the base in each of these logarithms, it is apparent that n= -1

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In the first logarithm n=-1 and in the second logarithm n= -3. When added n= -1 + -3= -4, therefore the third answer is obtained by multiplying the bases together in accordance with the rules of exponents.

The next two examples, which would fit the pattern would therefore be:

$$\log_{5^{-7}} 5^3, \log_{5^{-11}} 5^3 = \log_{\frac{1}{78125}} 125, \log_{\frac{1}{48828125}} 125$$

4.

$$\log_8 512, \log_2 512, \log_{16} 512 = \log_{2^3} 2^9, \log_{2^1} 2^9, \log_{2^4} 2^9$$

In the first logarithm n=3 and in the second logarithm n=1. When added n=3+1=4, therefore the third answer is obtained by multiplying the bases together in accordance with the rules of exponents.

The next three examples that would fit the pattern would therefore be:

$$\log_5 512, \log_9 512, \log_{14} 512$$

Let $\log_a x = c$ and $\log_b x = d$. Find the general statement that expresses in terms of c and d.

$\log_a x = c$ and $\log_b x = d$ then find $\log_{ab} x$.

One law of logarithms states that:

$$\log_a x + \log_b x = \log_{ab} x$$

Use the change of base formula:

$\log_a x = c$ then $a^c = x$ $\log_b x = d$ then $\log_b x = d$

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$\therefore \log_a c = \log x$ $\therefore \log_b d = \log x$

Take the Logarithm in base x:

$\therefore c \log_x a = \log_x x$ $\therefore d \log_x b = \log_x x$

$\therefore c = \dfrac{\log_x x}{\log_x a} = \dfrac{1}{\log_x a}$ $\therefore d = \dfrac{\log_x x}{\log_x b} = \dfrac{1}{\log_x b}$

Derived from $\log_a x + \log_b x = \log_{ab} x$ it can be stated that:

$$\frac{1}{\log_x a} + \frac{1}{\log_x b} = \frac{1}{(\log_x a + \log_x b)}$$

Using the change of base formula the following expression is derived:

$$\frac{1}{\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x}}$$

Substitute: $\log_a x = c$ and $\log_b x = d$:

$$\frac{1}{\frac{1}{c} + \frac{1}{d}}$$

Multiply both sides by cd:

$$\frac{cd}{c+d}$$

The general statement that expresses $\log_{ab} x$ in terms of c and d is:

$$\frac{cd}{c+d}$$

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Test the validity of your general statement using other values of a, b, and x.

$\log_a x = c$ and $\log_b x = d$

1. Example: a=2, b=4, x=8

$\log_2 8 = c$ and $\log_4 8 = d$

$\log_2 8 + \log_4 8 = \log_8 8 = 1$

Check with the general statement:

$$c = \log_2 8 = \log_{2^1} 2^3 = \frac{3}{1} = 3$$
$$d = \log_4 8 = \log_{2^2} 2^3 = \frac{3}{2} = 1.5$$

$$\frac{cd}{c+d} = \frac{(3)(\frac{3}{2})}{3 + \frac{2}{3}} = \frac{1}{1} = 1$$

General Statement Justified

2. Example: a=5, b=125, x=25

$\log_5 25 = c$ and $\log_{125} 25 = d$
$\log_5 25 + \log_{125} 25 = \log_{625} 25 = 0.5$

Check with the general statement:

$$c = \log_5 25 = \log_{5^{-1}} 5^2 = \frac{2}{1} = 2$$

$$d = \log_{125} 25 = \log_{5^3} 5^2 = \frac{2}{3} = 0.67$$

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$$\frac{cd}{c+d} = \frac{(2)(\frac{2}{3})}{2+\frac{2}{3}} = \frac{1}{2} = 0.5$$

General Statement Justified

3. Example: a=1000, b= $\frac{1}{10}$, x=10

$\log_{10000} 10 = c$ and $\log_{\frac{1}{10}} 10 = d$

$$\log_{10000} 10 + \log_{\frac{1}{10}} 10 = \log_{10000} 10 = \frac{1}{3}$$

Check with the general statement:

$$c = \log_{10000} 10 = \log_{10^4} 10^1 = \frac{1}{4} = 0.25$$

$$d = \log_{\frac{1}{10}} 10 = \log_{10^{-1}} 10^1 = \frac{1}{-1} = -1$$

$$\frac{cd}{c+d} = \frac{(\frac{1}{4})(-1)}{\frac{1}{4}-1} = \frac{1}{3} = 0.33$$

General statement justified.

Discuss the scope and/ or limitations of a, b, x.

The limitations of logarithms are, as previously stated:

a>0, a$\neq$ 1, b>0

therefore the limitations for this question are as follows:

a>0, b>0, a≠1, b≠1, x>0

To check for validity of this statement:

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Example: a=-2, b=2, x=4

$\log_{-2} 4 = c$ and $\log_2 4 = d$

$\log_{-2} 4 + \log_2 4 = \log_{-4} 4 = impossible$

It is impossible to have a negative power in this function, in this case a=-2. With these numbers:

$\log_{2^n} 2^n$, n>0.

The same applies for b, that a>0.

Example 2: a=10, b=1, x=100

$\log_{10} 100 = c$ and $\log_1 100 = d$

$\log_{10} 100 + \log_1 100 = impossible$

n=0 as $\log_{10^0} 10^1$, which is impossible because it forced division to occur between that number and $\frac{1}{0}$ = error/ not possible

The example applies for a, that b≠1.

Example 3: a=4, b=8, c=-8.

$\log_4 - 8 = c$ and $\log_8 - 8 = c$

$\log_4 - 8 + \log_8 - 8 = impossible$

As seen in example 1, it is impossible to have a negative power in one of these functions.

With these numbers: $\log_{2^8} 2^k$, k>0.

As a>0 and b>0, the product x should always be greater than 0, therefore x>0.

In summary:
a>0, b>0, a≠1, b≠1, x>0

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Explain how you arrived at your general statement.

First, the sequences given with observed. After determining that each sequence had a constant exponent, but the bases did not remain constant, I used the change of base formula. This allowed the logarithm to be rewritten in terms of logs written with other bases.

$\log_a x = c$ then $a^c = x$ $\log_b x = d$ then $\log_b x = d$

$\therefore \log_a c = \log x$ $\therefore \log_b d = \log x$

After observing the sequences and determining the nth term, the validity of the general formula was tested using other values of a, b, and x. The validity was proved through the formula:

$$\log_a x + \log_b x = \log_{ab} x$$

After taking the logarithms in base x:

$\therefore c = \dfrac{\log_x x}{\log_x a} = \dfrac{1}{\log_x a}$ $\therefore d = \dfrac{\log_x x}{\log_x b} = \dfrac{1}{\log_x b}$

Therefore,

$$\frac{1}{\log_x a} + \frac{1}{\log_x b} = \frac{1}{(\log_x a + \log_x b)}$$

Then it is necessary to use the change of base formula once again to get the following expression:

$$\frac{1}{\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x}}$$

Since variables were given, it was most likely that substiution would be needed to find the general formula:

$$\frac{1}{\frac{1}{c} + \frac{1}{d}}$$

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Consequently, after multiplying both sides by cd, the general formula is:

$$\frac{cd}{c+d}$$