Investigation

Mathematical Investigation

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IB Mathematics Portfolio Type I

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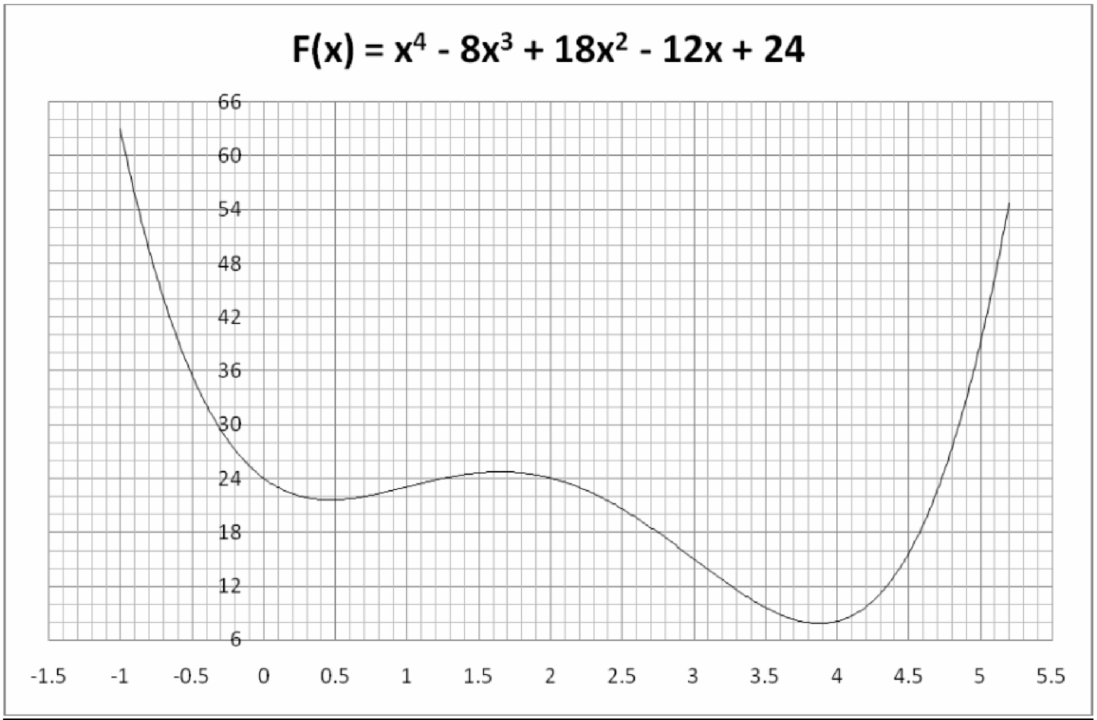
A quartic function with a "W" shape has two points of inflection, Q and R. In this investigation a line is drawn through Q and R to meet the quartic function again at P and S. The ratio PQ: QR: RS is to be investigated using specific examples to form a conjecture, and then examined formally to prove the findings.

1. **Graph the function f(x) = x^4 - 8 x^3 + 18x^2 - 12x + 24**



The above graph is made using "Microsoft Excel 2007" and the displayed function is
F(x) = $x^4 - 8x^3 + 18x^2 - 12x + 24$
Scale used: X: – 1.5 to 5.5
Y: 6 to 66

2. Find the coordinates of the points of inflection Q and R. Determine the points P and S, where the line QR intersects the quartic function again, and calculate the ratio PQ:QR:RS.

Planning:

- ✓ Find the points of inflection in the function by differentiating the function twice.
- ✓ Substituting the X values into the function to get Y co-ordinates.
- ✓ Get the equation of a line which passes through the inflection points, named Q and R respectively.
- ✓ Find out two points P and S, where f(x) intersects the line which passes through the inflection points (Q and R).
- ✓ Finding out the roots of the function, ignoring the found points of inflection and use these roots to find other intersection points for the function.
- ✓ To find the ratios we will need to find out the distances between the points using the properties of similar triangles.
- ✓ Find the ratio between the points PQ, QR, and RS. Then simplify the points to get some result.

We find out the points of inflection of the given function, where the concavity of the graph changes. When the concavity of the function changes, then at that point the second derivative of the function is equal to zero.

Differentiating the given function:
$F(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$
$F'(x) = 4x^3 - 24x^2 + 36x - 12$

Differentiate the function second time:
$F''(x) = 12x^2 - 48x + 36$

Solve the second derivative to get the values of x by equating it to zero.
=> $12x^2 - 48x + 36$
=> $x^2 - 4x + 3 = 0$
=> $(x - 3)(x - 1) = 0$
=> $x = 3$ and $x = 1$

Concavity of the function changes at $x = 3$ and $x = 1$. To find out the co-ordinate points of these inflection points, substitute x values in the given function and find Y-co-ordinates.

Point Q: -
$F(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$
$F(1) = 1^4 - 8(1)^3 + 18(1)^2 - 12(1) + 24 = 23$

Point R:

F(x) = $x^4 - 8x^3 + 18x^2 - 12x + 24$
F(3) = $3^4 - 8(3)^3 + 18(3)^2 - 12(3) + 24 = 15$

Therefore, the two points of inflection are:
Q (1, 23)
R (3, 15)

To find the equation of a line which passes through the points Q and R, we first name the function as G(x). Equation of a line which passes from any two points can be found by the formula.

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

We will now substitute the known values (co-ordinate points) and find out the equation of line.
$Y_1 = 15$
$Y_2 = 23$
$X_1 = 3$
$X_2 = 1$

Therefore, the equation is: $y - 15 = \frac{(23-15)}{(1-3)}(x - 3)$
Y = -4x + 27, where -4 is the slope of the line
=> G (x) = $-4x + 27$

Now our next step would be to find out the position of the points P and S. Points P and S are the intersection points of the functions f(x) and g(x). By equating them we will get the X co-ordinates of the points on which f (x) and g (x) intersect.

G(x) = F(x)
$-4x + 27 = x^4 - 8x^3 + 18x^2 - 12x + 24$
$x^4 - 8x^3 + 18x^2 - 8x - 3 = 0$

We will get 4 roots from the above equation; we already know two roots which are Q and R respectively. Therefore, to find out the other two roots we will use Synthetic Division. We can thus divide the function by the point Q with the X-value of 1 to acquire a cubic function from quartic:

$$\begin{array}{r|rrrrr} 1 & 1 & -8 & 18 & -8 & -3 \\ & & 1 & -7 & 11 & 3 \\ \hline & 1 & -7 & 11 & 3 & 0 \end{array}$$

The cubic is $x^3 - 7x^2 + 11x + 3 = 0$

Another root of the function can be found by dividing point R with X-value of 3 to get a quadratic function from cubic:

3	1	-7	11	3
		3	-12	-3
	1	-4	-1	0

The quadratic is $x^2 - 4x - 1 = 0$

Using the quadratic formula we can now find out the remaining two roots of the quartic.

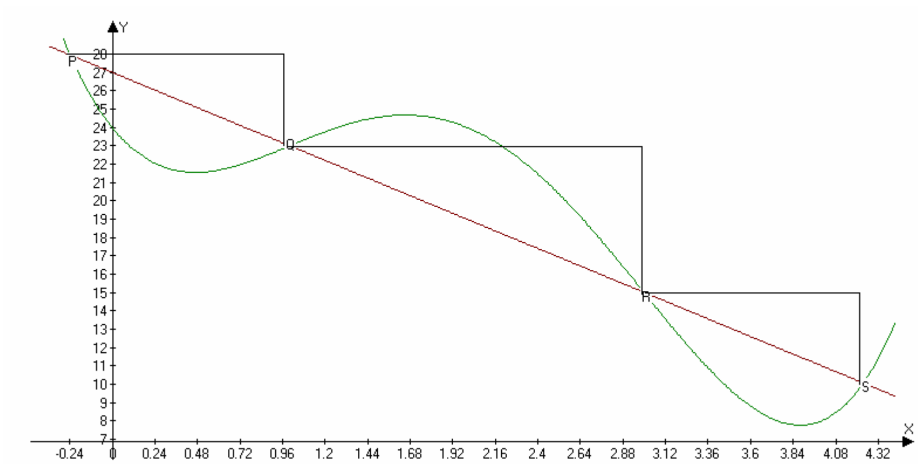
$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 + 4}}{2}$$

$$= 2 \pm \sqrt{5}$$

Roots of the function are therefore:
P: $x = 2 - \sqrt{5}$
Q: $x = 1$
R: $x = 3$
S: $x = 2 + \sqrt{5}$

The graph below shows 4 intersection points of the functions. The quartic function f(x) and a linear function g(x) have two inflection points Q and R, and the other intersection points are P and S respectively.



To find out the relationships between PQ, QR and RS lines we must make triangles created by the intersection points. All the three triangles made would be similar because all the three angles of these triangles are equal. This is because all these triangles have been created by horizontal and a vertical line of a function with a constant slope of a line. As the triangles are similar it is obvious that the ratios of its sides would be proportional which means that their ratios of sides for all the triangles would be equal. Therefore, using the sides of these similar triangles would be shorter way than to find out the distances between them and then finding out the ratio. The Catheti, which is the differences between the X and Y coordinates of the intersection points. We would use the known X values because the Y values have not been found.

The ratio of the points PQ: QR: RS equals to the ratio of differences between their X coordinates. An example is that the difference between the X coordinates of Q and P gives the segment PQ. We have to subtract the X value of the point P with the X value of point Q.

The ratio can be written as $Q_x - P_x$: $R_x - Q_x$: $S_x - R_x$

Inserting the X coordinate values to find the ratio $1-(2-\sqrt{5}) : 3-1 : 2+\sqrt{5}-3$

This simplifies to $(\sqrt{5}-1) : 2 : \sqrt{5}-1$

3. **Simplify this ratio so that PQ = 1 and comment upon your results.**

The ratio PQ: QR: RS found in Question 2 is $\sqrt{5}-1 : 2 : \sqrt{5}-1$

To make PQ = 1, divide the other two ratios by PQ

PQ: QR: RS = $\frac{\sqrt{5}-1}{\sqrt{5}-1} : \frac{2}{\sqrt{5}-1} : \frac{\sqrt{5}-1}{\sqrt{5}-1} =$

$1 : \frac{2}{\sqrt{5}-1} : 1$

Thus, we can conclude that PQ = RS and the ratios of PQ: QR and QR: RS are approximately 1.62, which is the golden ratio. We must get this golden ratio for question 4 to prove golden ratio as true.

4. **Choose another quartic function with a "W" shape and investigate the same ratios.**

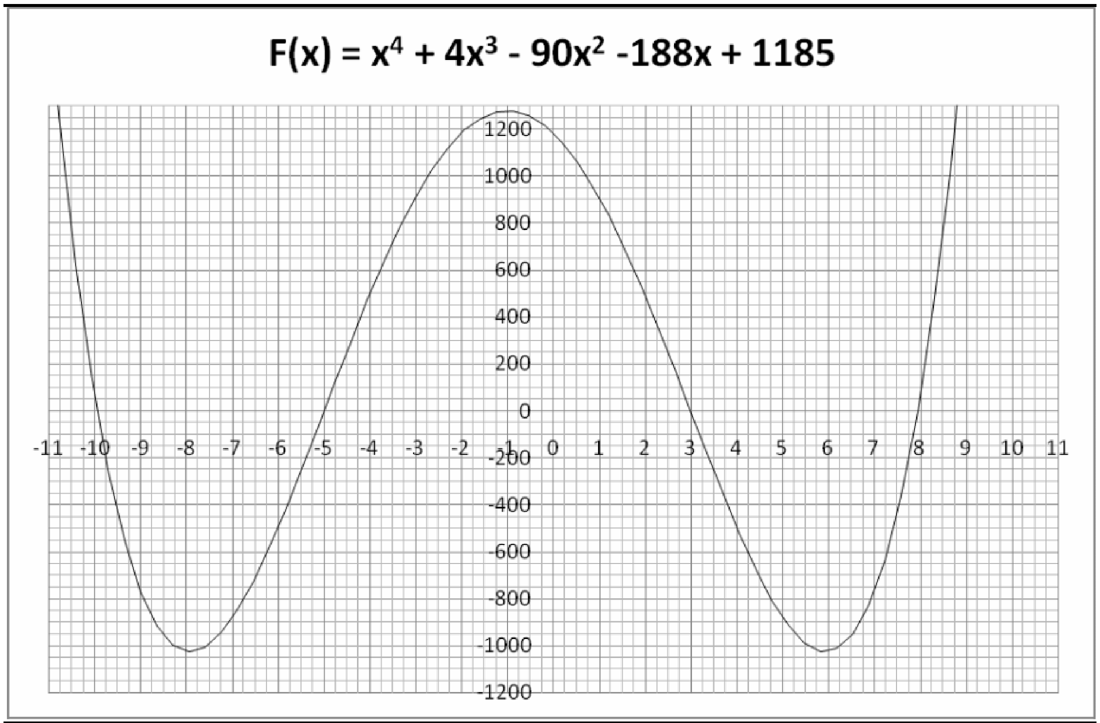
Planning:

- ✓ By inserting random formulas in the Graphic Display Calculator we get a "W" shaped quartic function.
- ✓ Find the points of inflection for the function by differentiating the function twice.
- ✓ Substitute the X values into F(x) to find Y co-ordinates.
- ✓ Find the equation for a line that pass through both inflection points, which would be Q and R respectively.
- ✓ Find two points P and S, where F(x) intersects the line which passes through the inflection points.
- ✓ Finding out the roots of the function, ignoring the found points of inflection and using them to get the other two intersection points.
- ✓ To get the ratios, we will have to find the distances between the points using properties of similar triangles. Find PQ, QR, and RS. Then simplify them.

Using several attempts of making random graphs on graphing display calculator, a following quartic function was found to be "W" shaped.

$f(x) = x^4 + 4x^3 - 90x^2 - $[illegible]$88x + $[illegible]$85$

The graph of the function is shown below:



This graph is drawn using "Microsoft Excel 2007".
Scales used: X: -11 to 11
Y: -1200 to 1300

Finding the inflection points, where the second derivative of the function equals to zero.
Differentiating the function to get the first derivative and then again to find the second derivative of the function:

$F(x) = x^4 + 4x^3 - 90x^2 - 188x + 1185$

$F'(x) = 4x^3 + 12x^2 - 180x - 188$

$F''(x) = 12x^2 + 24x - 180$

Setting the second derivative equal to zero,

$12x^2 + 24x - 180 = 0$

Dividing the equation by 6 to get:

$2x^2 + 4x - 30 = 0$

$2x^2 + 10x - 6x - 30 = 0$

$$2x(x+5) - 6(x+5) = 0$$
$$x = 3 \text{ and } x = -5$$

Substitute the X-values into F(x) to find Y-values, thus
Point Q:

$F(x) = x^4 + 4x^3 - 90x^2 - 188x + 1185$

$F(x) = (3)^4 + 4(3)^3 - 90(3)^2 - 188(3) + 1185 = 0$

Point R:

$\mathrm{F}(x) = x^4 + 4x^3 - 90x^2 - 188x + 1185$
$\mathrm{F}(x) = (-5)^4 + 4(-5)^3 - 90(-5) - 188(-5) + 1185 = 0$
Therefore points of inflection are Q (3, 0) and R (-5, 0)

Finding out the equation for a line that pass through the points of inflection (Q and R), name the function of the line as G(x)
Equation of line is given by the formula

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

$$y = \frac{0}{-8}(x + 3)$$

y = 0

$=> G(x) = 0$

Finding out the points of intersection between the two functions f(x) and g(x), this can be done by equating them. Set the functions equal to find intersection points:

F(x) = G(x)

$$x^4 + 4x^3 - 90x^2 - 188x + 1185 = 0$$

Using Synthetic Division we can find the other two roots of the quartic. Dividing the function by (x - 3) and then by ($x + 5$) to get the function in quadratic so to find the other roots.

3	1	4	-90	-188	1185
		3	21	-207	-1185
	1	7	-69	-395	0

The cubic is therefore $x^3 + 7x^2 - 69x - 395 = 0$

Second division by the root of point Q (-2) will give us a quadratic

-5	1	7	-69	-395
		-5	-10	395
	1	2	-79	0

The quadratic is therefore $x^2 + 2x - 79 = 0$

Solving this with the quadratic formula gives us the two remaining roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{320}}{2}$$

$$= -1 \pm \sqrt{80}$$

Roots of the function are therefore written below:
P: $x = -1 - \sqrt{80}$
Q: $x = -5$
R: $x = 3$
S: $x = -1 + \sqrt{80}$

As it is a straight line, we can directly find out the ratios of x values.

The ratio of PQ: QR: RS equals $-5-(-1-\sqrt{80}):3+5: \ -1+\sqrt{80}- \ 3$

This simplifies to $\sqrt{80}-4:8:\sqrt{80}-4$

Simplify the ratios so that PQ = 1 and comment on the results. To make PQ = 1, divide all the ratios by the ratio of PQ which is $\sqrt{5}-1$.

PQ: QR: RS ≈

$$\frac{\sqrt{80}-4}{\sqrt{80}-4}:\frac{8}{\sqrt{80}-4}:\frac{\sqrt{80}-4}{\sqrt{80}-4}$$

$$= 1: \ 1.62 : 1$$

5. **Form a conjecture and formally prove it using a general quartic function**

Instead of differentiating the general quartic function twice, we can start by selecting two points of inflection and integrating them twice to get the function.

Let the first point of inflection point Q be the origin and may the second point R be defined by the two variables, a and f(a). This is the second point of inflection of the general quartic function. The co-ordinates of the points of inflections are point Q is (0, 0) and point R is (a, f (a)) respectively.

Roots of the second derivative for a particular function are its inflection points. The roots are 0 and a in the given example. By multiplying $(x-0)$ and $(x-a)$ with one other we obtain the function for the second derivative, as these two are the roots of the second derivative function.

After integrating the second derivative twice we get the original quartic function. This quartic could be used to deduce the co-ordinates of point R in the terms of a. Once both the points have been found, we can find out the equation of the line which passes through them. The equation of the line can be derived when the slope is known. The slope of the line can be found by simply dividing the Y-co-ordinate of the point R by its own X coordinate because the other co-ordinate we have passes through the origin. Equating the equation of line and the quartic function with each other will give us all the four points of inflection P, Q, R, and S respectively.

By using synthetic division and known roots we can find the equation of the quartic into quadratic as done previously, which can be solved using quadratic formula. The X coordinates of the points P and S which are remaining can be found from the roots of the quadratic. Therefore, the ratio of PQ: QR: RS can be found.

$$f''(x) = (x-0)(x-a)$$

$$= x^2 - xa$$

Integrating the function twice will give us the original quartic function.

$$f'(x) = \int x^2 - xa = \frac{x^3}{3} - \frac{ax^2}{2} + C$$

$$f(x) = \int \frac{x^3}{3} - \frac{ax^2}{2} + C$$

$$= \frac{x^4}{12} - \frac{ax^3}{6} + Cx + d$$

We had given the initial condition of point of inflection at origin which means f(0) = 0, and thus constant "c" can be found.

$$f(0) = \frac{0^4}{12} - \frac{a0^3}{6} + (C \times 0) + d = 0$$

=> The constant is found which is $d = 0$.

The quartic function can now be reduced to

$$f(x) = \frac{x^4}{12} - \frac{ax^3}{6} + Cx$$

To find the point R, we need to find the value of f (a) which is the value of Y co-ordinate.

$$f(a) = \frac{a^4}{12} - \frac{aa^3}{6} + Ca$$

$$= \frac{a^4}{12} - \frac{a^4}{6} + Ca$$

$$= -\frac{a^4}{12} + Ca$$

$$\text{Therefore, point R is at } \left(\mathrm{a}, -\frac{\mathrm{a}^4}{12} + \mathrm{Ca}\right)$$

Equation for a line which passes through the points Q and R can be given by the formula
$g(x) = mx + b$
Where b is equal to zero because the line passes through the origin (point Q) and m is the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{y_2 - 0}{x_2 - 0}$$
$$= \frac{-\frac{\mathrm{a}^4}{12} + \mathrm{Ca}}{a}$$
$$= -\frac{a^3}{12} + C$$
$$g(x) = mx + b = \left(-\frac{a^3}{12} + C\right)x + 0$$
$$g(x) = -\frac{a^3 x}{12} + Cx$$

Finding out the intersection points between the quartic function and line by simply equating them.

$$-\frac{a^3 x}{12} + Cx = \frac{x^4}{12} - \frac{\mathrm{a}x^3}{6} + \mathrm{C}x$$

$$-a^3x + 12Cx = x^4 - 2ax^3 + 12Cx$$

$$-x^4 - a^3x + 2ax^3 = 0$$

$$x^4 - 2ax^3 + a^3x = 0$$

Using synthetic division we can find the remaining roots by dividing the quartic by that root.

a	1	-2a	0	a^3	0
		a	$-a^2$	0	0
	1	-a	$-a^2$	0	0

The function is thus reduced to a cubic $x^3 - ax^2 - a^2x = 0$.
Divide further by the second root, $x = 0$.

0	1	-a	$-a^2$	0
		0	0	0

1 -a -a^2 0

The quartic is reduced to a quadratic function $x^2 - ax - a^2 = 0$
We can find the roots of the function quadratic equation

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{a \pm \sqrt{(-a)^2 + 4a^2}}{2}$$
$$= \frac{a \pm \sqrt{5a^2}}{2}$$
$$= \frac{a \pm a\sqrt{5}}{2}$$

The ratio of PQ: QR: RS is therefore

$$\frac{a \pm a\sqrt{5}}{2} : a : \frac{a \pm a\sqrt{5}}{2}$$

Divide by [illegible] to set QR equal to one:

$$\frac{0.5a(1 \pm \sqrt{5})}{a} = \frac{1 \pm \sqrt{5}}{2}$$

The ratio therefore is:

$$\frac{1 \pm \sqrt{5}}{2} : 1 : \frac{1 \pm \sqrt{5}}{2}$$

If calculated, it simplifies to

-0.62: [illegible].-0.62 [illegible].62: [illegible].[illegible].62

6. **Extend this investigation to other quartic functions that are not strictly of a "W" shape.**

All quartics of a W shape have two distinct inflection points but these properties do not apply to all quartic functions. Four types of quartics have been set up and are written below:

- ✓ Quartics with two distinct and real inflection points (W shape)
- ✓ Quartics without any points of inflection
- ✓ Quartics with one point of inflection
- ✓ Quartics with two points of inflection

The first type of quartic is W shaped if coefficient of x^4 is negative, which was proven to contain the Golden Ratio previously in question 3 and 4. Quartics with no inflection points, such as f(x) = x^4, cannot be used to determine the ratio because there would be no line between the inflection points. To find the conditions for such a quartic, a general quartic can be considered for example

If $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, concavity of the function will changes when the second derivative of the function is zero. Differentiate twice to get the quadratic:

$$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$$
$$f''(x) = 12ax^2 + 6bx + 2c = 0$$

By the quadratic formula we can find the X values of points of inflection:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-6b \pm \sqrt{36b^2 - 96ac}}{24a}$$

We will find one inflection point be present if the roots of the quadratic are equal, which is the case when discriminant is zero. From the above formula, we find that the quartic has one inflection point when$36b^2 - 96ac = 0$. We would not be able to find the ratio as point R will be missing in the function.

The last type of quartic does posses two distinct and real inflection points. From the definition we know that points of inflection occur when $f''(x) = 0$, and we also know that function is a straight line when $f'(x) = 0$. Points of inflection will be at X values where $f'(x) = f''(x)$.

Given a general quartic $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, and using previously arrived first and second derivatives, we arrive at this condition:

$$4ax^3 + 3bx^2 + 2cx + d = 12ax^2 + 6bx + 2c$$
$$4ax^3 + 12ax^2 + 3bx^2 + 6bx + 2cx + 2c + d = 0$$

Any quartic which satisfies this equation will thus have at least one real inflection point.

In case of inflection points, the ratio PQ: QR: RS will be the same as in any "W" shaped quartic. This is because the general conjecture from question 5 applies to all quartics with two distinct inflection points.

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