This portfolio is an attempt at deriving and examining the scope and limitations of a general statement that can approximate the area under a curve using trapezoids. Generally, calculus – specifically the method of integration, is used to find the exact area under a curve. Although this method will be explored in comparison later in the portfolio, this investigation deals mainly with investigating a method to approximate the area using high school level math. First, this portfolio will attempt to derive a general statement that will give an approximation of the area under the curve of any function in any closed interval using n trapezoids. Then, by applying the formula to sample functions, the answers given can be compared to the integral answers, allowing an examination into the accuracy of the trapezoid method of approximation. Lastly, by examining different behaviors of a graph, this portfolio will investigate the cause of any inaccuracies in this method.

The graph: $g(x) = x^2 + 3$ is given:

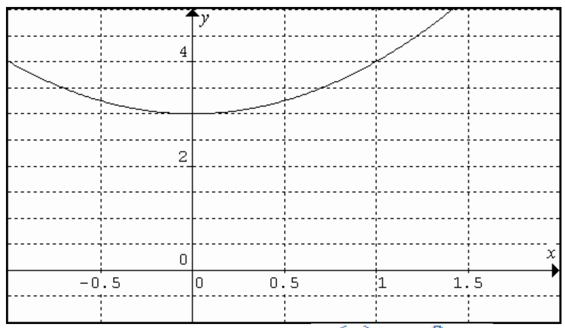

Figure 1 – Graph of $g(x) = x^2 + 3$

Using *two* trapezoids mapped onto the curve in the domain $D: \{x | 0 \leq x \leq 1, x \in \mathbb{R}\}$, the area under the curve in that domain can be approximated as the sum of the areas of the two trapezoids.

In order to map the trapezoids onto the graph, one must first divide the domain by the number of trapezoids being used, in order to find the height of the trapezoids (which is equivalent to each other). It should be noted that the height is not a vertical distance in this case, but the distance between the two parallel sides of a trapezoid. With the general case $D: \{x | a \leq x \leq b, x \in \mathbb{R}\}$, height can be calculated with the formula $\frac{b-a}{n}$ where n is the number of trapezoids.

In this case, the height will work out to:

$\frac{1-0}{2} = 0.5.$

Thus, the first (from the left) trapezoid will be mapped from $g(0)$ to $g(0.5)$, and the second trapezoid from $g(0.5)$ to $g(1)$. The first trapezoid will have vertices at $(0, 0), (0.5,0), (0, g(0)), (0, g(0.5))$ and the second trapezoid will have vertices at $(0.5, 0), (1,0), (0, g(0.5)), (0, g(1))$:

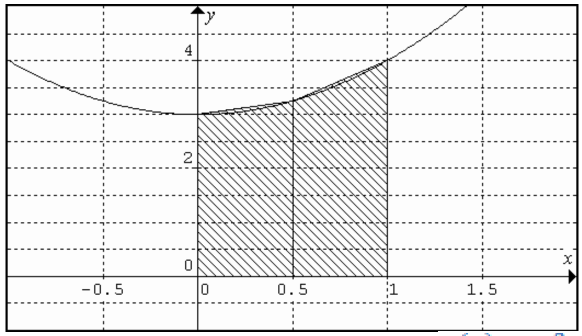

Figure 2 – Two trapezoids mapped onto $g(x) = x^2 + 3$

The formula to calculate the area of a trapezoid is given as $A = \frac{1}{2}h(a + b)$ where h is the height, and a and b are the lengths of the parallel sides.

For the first trapezoid, the area is calculated to be:

$$A_1 = \frac{1}{2}(0.5)[g(0) + g(0.5)] = \frac{1}{4}(3 + 3.25) = 1.5625\ units^2$$

And the second trapezoid:

$$A_2 = \frac{1}{2}(0.5)[g(0.5) + g(1)] = \frac{1}{4}(3.25 + 4) = 1.8125\ units^2$$

And the total area being:

$A_{total} = A_1 + A_2 = 1.5625\ units^2 + 1.8125\ units^2 = 3.38\ units^2$

For the same function, examine the calculation when the same method is attempted with 5 trapezoids:

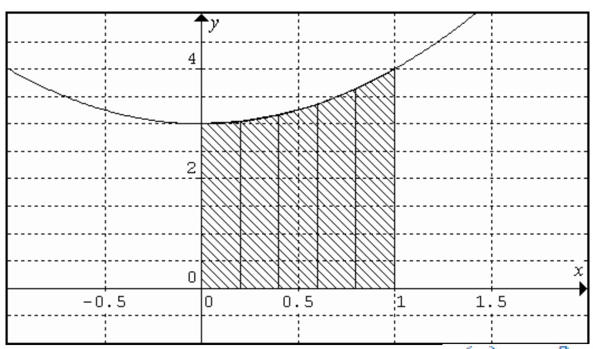

Figure 3 – Five trapezoids mapped onto $g(x) = x^2 + 3$

The height will work out to:

$\frac{1-0}{5} = 0.2$

Thus, the trapezoids will be mapped between $g(0)$ and $g(1)$ with equivalent heights of 0.2.

Using the formula for area of a trapezoid, the total area is calculates as the sum of 5 separate area calculations, one for each trapezoid. They all share the common factor of $\frac{1}{2}$ and 0.2, what differentiates them will be the g(x) values. The first trapezoid will have those values as [g(0) + g(0.2)], the second as [g(0.2) + g(0.4)], and so on until the 5th trapezoid as [g(0.8) + g(1)]. This gives the following calculation:

$$A_{total} = A_1 + A_2 + A_3 + A_4 + A_5$$

$$A_{total} = \frac{1}{2}(0.2)[g(0) + g(0.2)] + \frac{1}{2}(0.2)[g(0.2) + g(0.4)] + \frac{1}{2}(0.2)[g(0.4) + g(0.6)] + \frac{1}{2}(0.2)[g(0.6) + g(0.8)] + \frac{1}{2}(0.2)[g(0.8) + g(1)]$$

Factoring out $\frac{1}{2}$ and (0.2):

$$A_{total} = \frac{1}{2}(0.2)[g(0) + g(0.2) + g(0.2) + g(0.4) + g(0.4) + g(0.6) + g(0.6) + g(0.8) + g(0.8) + g(1)]$$

$$A_{total} = (0.1)(3 + 3.04 + 3.04 + 3.16 + 3.16 + 3.36 + 3.36 + 3.64 + 3.64 + 4) = 3.34\ units^2$$

The area approximated by using 5 trapezoids differs in value to the approximation using only 2 trapezoids. Examine the following diagrams of the approximated area using 1-8 trapezoids:

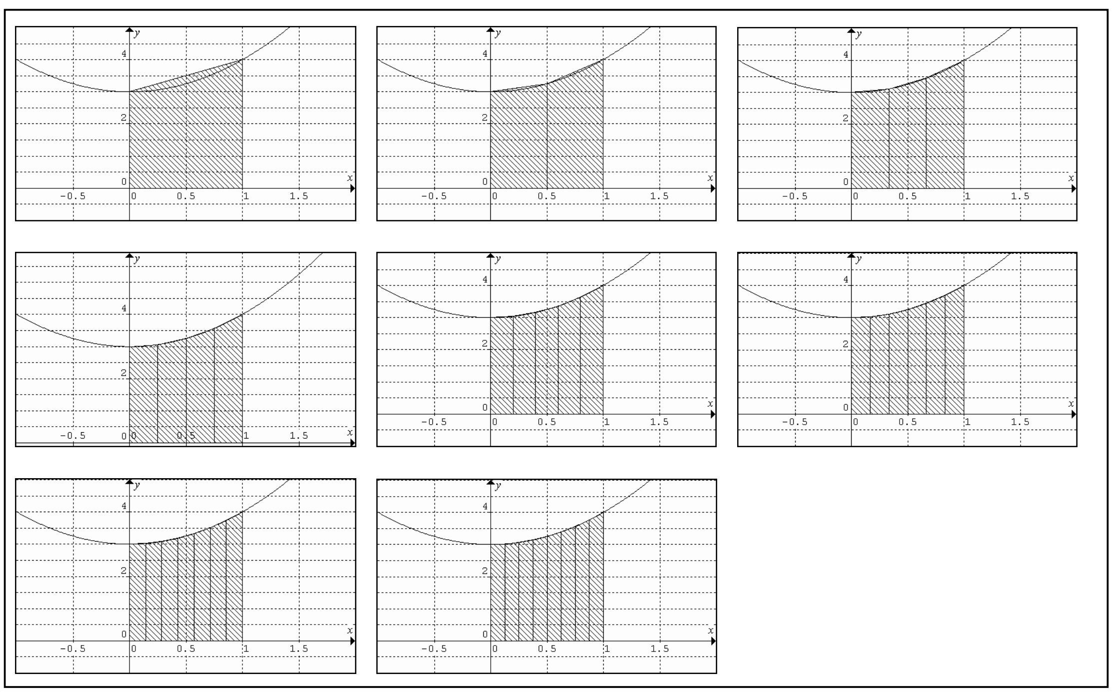

Diagram 1 – 1-10 trapezoids mapped onto $g(x) = x^2 + 3$

In a chart, the approximated areas (calculated using the program Graphmatica) are:

# of Trapezoids Used	Approximated Area (units^2)
1	3.50
2	3.375
3	3.514
4	3.348
5	3.34
6	3.338
7	3.3367
8	3.3359

Diagram 2 – Approximated Areas (left unrounded to showcase increasingly gradual differences)

One can notice with this data that as the area is approximated with an increasing number of trapezoids, the approximated area approaches a limit similar to how a function would approach an asymptote.

Comparing the calculation used to approximate the area with 5 trapezoids to the calculation with 2 trapezoids, it is possible to come up with a general expression that can be used for the function $g(x)$ in any domain, with any number of trapezoids. In doing that, first examine the diagram below, which shows how to map n trapezoids under the curve for the function $g(x) = x^2 + 3$, in the domain $D: \{x | 0 \leq x \leq 1, x \in \mathbb{R}\}$:

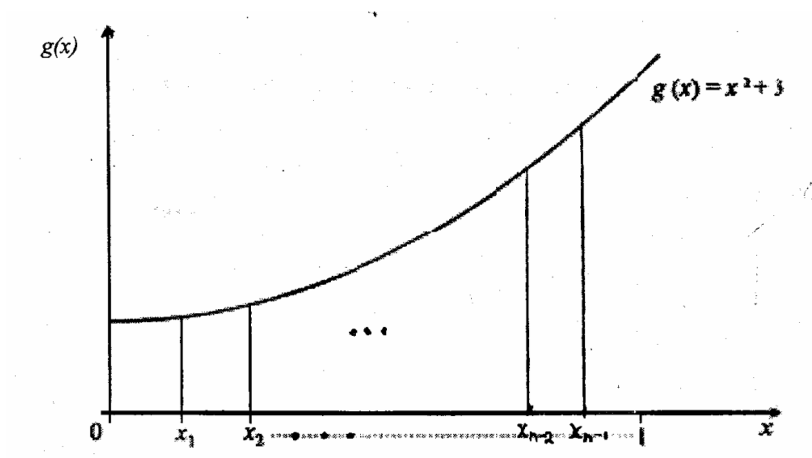

Diagram 3 – Illustration of how to map n trapezoids onto $g(x) = x^2 + 3$

To begin, one needs to find the area of each of the trapezoids that will be mapped onto this function. That will be done by manipulating the area of a trapezoid formula of $A = \frac{1}{2}h(a + b)$ for this specific case. h can be replaced with $\frac{1-0}{n}$, where n is the number of trapezoids used. a+b can be replaced with: $g(0) + g(x_1)$ for the first trapezoid, $g(x_1) + g(x_2)$ for the second trapezoid, $g(x_2) + g(x_3)$ for the third and so on. Because x_n is *1*, the second to last a+b replacement will be $g(x_{n-2}) + g(x_{n-1})$, and the last one $g(x_{n-1}) + g(1)$.

This gives the following formulas for the area of each trapezoid:

$A_1 = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(0) + g(x_1)]$

$A_2 = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_1) + g(x_2)]$

$A_3 = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_2) + g(x_3)]$

...

$A_{n-1} = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_{n-2}) + g(x_{n-1})]$

$A_n = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_{n-1}) + g(1)]$

In adding the areas of multiple trapezoids, the separate area formulas for each of them will be terms in a calculation whose sum gives total area, as was demonstrated in the calculation for 5 trapezoids. Thus:

$$A_{total} = A_1 + A_2 + A_3 + \ldots + A_{n-1} + A_n$$

$$A_{total} = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(0) + g(x_1)] + \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_1) + g(x_2)] + \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_2) + g(x_3)] + \ldots \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_{n-2}) + g(x_{n-1})] + \frac{1}{2}\left(\frac{1-0}{n}\right)[g(x_{n-1}) + g(1)]$$

Which can be factored by taking out the values of $\frac{1}{2}$ and $\left(\frac{1-0}{n}\right)$:

$$A_{total} = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(0) + g(x_1) + g(x_1) + g(x_2) + g(x_2) + g(x_3) \ldots + g(x_{n-2}) + g(x_{n-1}) + g(x_{n-1}) + g(1)]$$

Noticing the terms $g(x_k)$ – where k is any whole number – each appear twice, the expression can be further simplified:

$$A_{total} = \frac{1}{2}\left(\frac{1-0}{n}\right)[g(0) + 2g(x_1) + 2g(x_2) + 2g(x_3) \ldots + 2g(x_{n-2}) + 2g(x_{n-1}) + g(1)]$$

The above is now the general expression for finding the area under the curve in the function $g(x) = x^2 + 3$, from $x = 0$ to $x = 1$ using n trapezoids.

In order to develop from that a general statement that will estimate the area under any curve where $y = f(x)$, and in the general domain of $D: \{x | a \leq x \leq b, x \in \mathbb{R}\}$ using n trapezoids, one can modify the formula to accept the variables a instead of 0 and b instead of 1, as well as replacing $g(x)$ with $f(x)$, giving:

$$A_{total} = \frac{1}{2}\left(\frac{b-a}{n}\right)[f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(b)]$$

As examples, this investigation will find approximated areas for the following 3 functions using the general formula derived above, in the domain $D: \{x | 1 \leq x \leq 3, x \in \mathbb{R}\}$, using 8 trapezoids:

Example Function 1	Example Function 2	Example Function 3
$f(x) = \left(\frac{x}{2}\right)^{\frac{2}{3}}$	$f(x) = \frac{9x}{\sqrt{x^3 + 9}}$	$f(x) = 4x^3 - 23x^2 + 40x - 18$

Diagram 4 – Three example functions to be investigated

1)

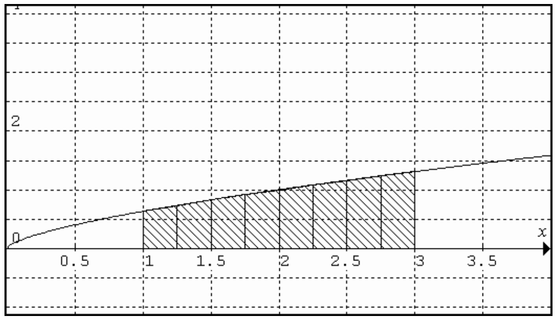

Figure 4 – Example Function 1 mapped with 8 trapezoids

$$A_{total} = \frac{1}{2}\left(\frac{3-1}{8}\right)[f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + 2f(x_6) + 2f(x_7) + f(b)]$$

$$A_{total} = \frac{1}{8}\left[\left(\frac{1}{2}\right)^{\frac{2}{3}} + 2\left(\frac{\frac{5}{4}}{2}\right)^{\frac{2}{3}} + 2\left(\frac{\frac{3}{2}}{2}\right)^{\frac{2}{3}} + 2\left(\frac{\frac{7}{4}}{2}\right)^{\frac{2}{3}} + 2\left(\frac{2}{2}\right)^{\frac{2}{3}} + 2\left(\frac{\frac{9}{4}}{2}\right)^{\frac{2}{3}} + 2\left(\frac{\frac{5}{2}}{2}\right)^{\frac{2}{3}} + 2\left(\frac{\frac{11}{4}}{2}\right)^{\frac{2}{3}} + \left(\frac{3}{2}\right)^{\frac{2}{3}}\right]$$

$$A_{total} = \frac{1}{8}(15.8402) = 1.98\,units^2$$

2)

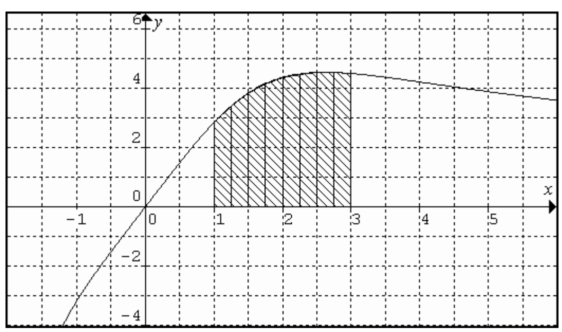

Figure 5 – Example Function 2 mapped with 8 trapezoids

$using\ the\ same\ method$

$$A_{total} = 8.25\,units^2$$

3)

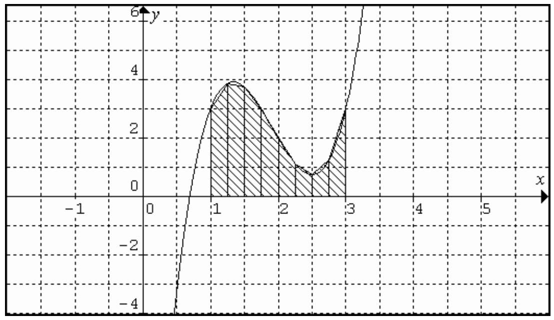

Figure 6 – Example Function 3 mapped with 8 trapezoids

using the same method of substitution as in a) ...

$A_{total} = 4.69\ units^2$

To measure the accuracy of the approximations generated by using 8 trapezoids, this investigation will compare its results with the mathematically exact procedure of integrating the functions. The integrals will be calculated using a Ti-84 Plus Calculator's fnInt function.

1) $\int_1^3 \left(\frac{x}{2}\right)^{\frac{2}{8}} dx$ 2) $\int_1^3 \left(\frac{9x}{\sqrt{x^8+9}}\right) dx$ 3) $\int_1^3 (4x^3 - 23x^2 + 40x - 18)\, dx$

1) $A = 1.98\ units^2$ 2) $A = 8.26\ units^2$ 3) $A = 4.67\ units^2$

Comparing the results:

Function	using the general statement with 8 trapezoids	using the method of integration
$f(x) = \left(\frac{x}{2}\right)^{\frac{2}{8}}$	$1.98\ units^2$	$1.98\ units^2$
$f(x) = \frac{9x}{\sqrt{x^8+9}}$	$8.25\ units^2$	$8.26\ units^2$
$f(x) = 4x^3 - 23x^2 + 40x - 18$	$4.69\ units^2$	$4.67\ units^2$

Diagram 5 – Comparing Trapezoid Method with Integration

One can notice that the approximations using the area of 8 trapezoids are very close to the actual area under the curves. For example function 1, it was equivalent (when both answers are taken to 3 significant digits). For example function 2, it was off by 1/100th. For example function 3, it was off by 2/100th. This begs the question, why are there greater differences in accuracy depending on the function?

Looking at the graphs, it becomes clear as to the reason. The accuracy of the trapezoid method is dependent on the behavior of the graph within the domain in which the trapezoid method is being applied. When the graph is increasing at a semi-steady rate, as in the first function, the trapezoid method excels in approximating the area under the curve with accuracy. It can be concluded then that the closer the slope of a curve is to being constant, the more accurate the approximation. However, when the domain contains a maximum or minimum point of any kind within the domain specified, the accuracy decreases because the slope of the graph changes from negative to positive, gradually, but the trapezoid can only make abrupt changes. The second function has one absolute max point within the specified domain, and the trapezoid method's accuracy was 1/100th off. The third function has both its local max and min points in the domain specified, and the accuracy was twice as deviant as the second function's was.

In theory, this inaccuracy could be addressed by using a greater number of trapezoids to approximate the area. Because the inaccuracy stems from the linearity of a trapezoid's edge – the smaller and more numerous the trapezoids, the more closely they can be mapped to the curvity of the function and the more accurate the approximation. What would make the result no longer an approximation but the actual answer could then be theorized to be the use of an infinite number of trapezoids. Manipulating the derived formula to accept the use of an infinite number of trapezoids gives:

$$A_{total} = \frac{1}{2}\left(\frac{b-a}{\infty}\right)[f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) \dots + 2f(x_\infty) + f(b)]$$

To simplify this, first examine the terms $2f(x_k)$ where k is any whole number. They all share a factor of 2, and the number of terms there is dependent on how many trapezoids are being used. Thus, factoring out the 2 and replacing the rest with a summation to infinity results in the following formula:

$$A_{total} = \frac{1}{2}\left(\frac{b-a}{\infty}\right)\left[f(a) + 2\left(\sum_{k=1}^{\infty} f(x_k)\right) + f(b)\right]$$

This formula however, is useless when one examines how to calculate a summation to infinity of a sequence. Inspect the formula:

$$S_\infty = \frac{a}{1-r},$$

a is the first term in the sequence (also equivalent to $f(x_1)$)and r is the rate of change between terms. Since the rate of change is neither constant nor calculable without calculus for an infinite number of infinitely thin trapezoids, this theoretical formula is useless. However, the same principal of using an infinite number of polygons to measure the area under a curve is used in the mathematical process of integration, which is the accepted way of finding the definite value of the area under a curve.

Returning to approximating the area under a curve using a finite number of trapezoids, this investigation has examined how a max/min point within the specified domain affects the accuracy of the approximation. Having done that, the next step is exploring how other behaviors of a graph can influence the accuracy of the approximations. For example, can this method of approximation be used when the domain specified has the graph approaching a vertical asymptote?

Take the function $f(x) = \frac{1}{(x-1)^2}$:

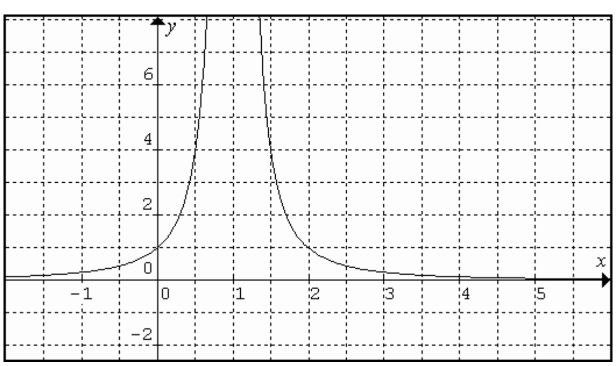

Figure 7 – Graph of $f(x) = \frac{1}{(x-1)^2}$

The trapezoid method of approximating the

area under the curve in the domain $D: \{x | 1 \leq x \leq 3, x \in \mathbb{R}\}$ will fail, because the lower limit of the interval is unbounded. Because there is no value to begin the area calculation, it is essentially infinite and thus an incalculable value.
However, upon investigation, the method of integration: $\int_1^3 \left(\frac{1}{(x-1)^2}\right) dx$ also produces an error when trying to calculate the area under the curve. Thus, it can be concluded that this inability to measure the area in a domain that has a vertical asymptote in it not a limitation on the trapezoid method of approximation itself, but simply a mathematically impossibility. From this conclusion, it can also be deduced that trying to find the area under a curve where the domain is composed of one or more open intervals – for example $D: \{x | a < x < b, x \in \mathbb{R}\}$, will fail for the same reasons.

As another investigation, take the trigonomic function $f(x) = \cos x$:

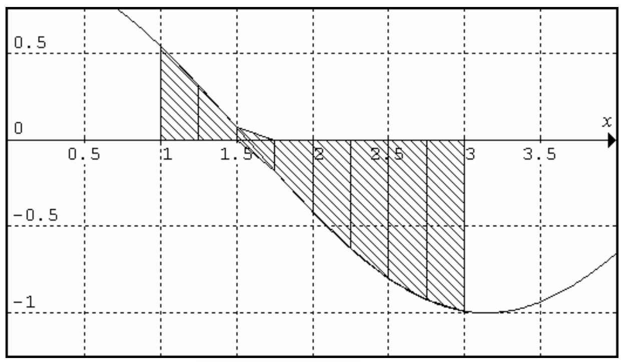

Figure 8 – Graph of $f(x) = \cos x$ mapped with 8 trapezoids

The area approximated as:

$$A_{total} = \frac{1}{2}\left(\frac{b-a}{n}\right) [f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(b)]$$

$$A_{total} = \frac{1}{2}\left(\frac{3-1}{8}\right)\left[\cos 1 + 2\cos\frac{5}{4} + 2\cos\frac{3}{2} + 2\cos\frac{7}{4} + 2\cos 2 + 2\cos\frac{9}{4} + 2\cos\frac{5}{2} + 2\cos\frac{11}{4} + \cos 3\right]$$

$$A_{total} = -0.70\ units^2$$

The actual area:

$$A_{total} = \int_1^3 \cos x$$

$$A_{total} = -0.70\ units^2$$

The behavior of the graph is trigonomic. Except for oscillation, the behavior of the graph is similar to a third degree polynomial function, such as the one in Figure 8. Thus, one can assume that the accuracy will be similar in how the local max/min's within the specified domain. The other behavior which this function shows that has not been explored is how the method of approximation works when the graph goes below the x axis. In examining the answers generated by both the trapezoid method and the integration method being negative, it can be concluded that area "above the curve" when it is below the x axis will be counted as negative area.

The method of approximation in this case would not encounter any deviances if the interval of the individual trapezoids happens to correspond with the x intercept, but this is rarely the case. As shown in Figure 8, the third trapezoid from the left that is mapped onto the function, an abnormality occurs when the x intercept lies within a mapped trapezoid. Roughly a third of the trapezoid is above the x axis (has positive area) and the rest under the x axis (has negative area). Thus, one can see that when mapping the trapezoid the vertex $f(x_2)$ will not connect with the vertex at $f(x_3)$, but the one on the x axis itself, x_3. Consequently, $f(x_3)$ is connected with x_2, enclosing the polygon. While this abnormality occers in trying to graph the function, no abnormality occurs in the mathematics. Examining the third trapezoid, its area is defined as $A_3 = \frac{1}{2}\left(\frac{3-1}{8}\right)\left[\cos\frac{3}{2} + \cos\frac{7}{4}\right]$, which works out correctly as the area above the curve subtracted from the area under the curve.

Although this case provided an approximation that was accurate, examining the graphical abnormality shows a significant source of error. Zoomed in on the graph:

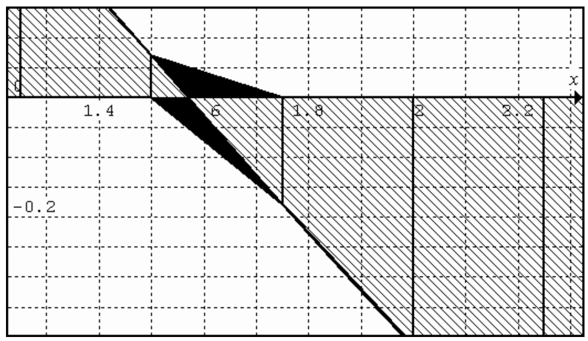

Figure 9 – Zoom in on Trapezoid 3 of Figure 10

The blacked out portions show where the trapezoid's area has been included despite being significantly deviant to the function itself. Thus, if the domain included multiple x intercepts, the inaccuracies would mount. This applies to all functions, and is not specific to trigonomic ones. It is notable however, that using more trapezoids will reduce the error in the same way that it will reduce the error for inaccuracies caused by max/min points within the domain.

Thus, this investigation has found 2 main sources of error influencing the trapezoid method of approximating the area under a curve: if the domain includes local or absolute max/min points, and if the domain includes x intercepts.

In conclusion, this study has determined that it is possible to approximate the area under any curve without using the method of integration by mapping trapezoids under the curve. A formula that does exactly that was derived, and this method's scope, limitation and accuracy were explored by applying the formula to different functions. In defining the scope of the trapezoid method, this investigation found it impossible to find the area under a curve when the graph approached a vertical asymptote within the specified domain. The limitations of the trapezoid method were determined to lie in 2 separate areas: the number of trapezoids used and the behavior of the graph (specifically if the specified domain included max/min points and x intercepts) both affected the accuracy of the approximation. In trying to eliminate inaccuracy, this study found a theoretical method to find the exact value of the area under a curve, but it was unusable due to a indeterminable variable. The theory though, corresponded with the theory behind the method of integration in that they both used an infinite number of polygons mapped under the curve.

Appendix:

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Technology Used:

Graphmatica Free Trial by kSoft
TI 84-Plus Silver Edition Graphing Calculator
Microsoft Office Word 2007

IB Math SL Type 1 Internal Assessment:
Shady Areas

By Calvin Ho
November 24th, 2009