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Investigating Matrix Binomials

Introduction to Matrix Binomials

Matrix Binomials can be defined as a type of a 2 by 2 matrix. Generally speaking, matrix binomials come in the form $M = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$. These matrix binomials can be defined as the sum of two component matrices. One component should be known as the positive matrix. All elements within the positive matrix have the same positive value. The other part should be called as the negative matrix. All elements within the negative matrix have the same magnitude but the top right and bottom left elements have a negative value. The overall goal of this project is to investigate the properties of these matrix binomials in relation to its positive and negative matrix components.

The first step would be investigating the positive and negative matrix components separately as they are the simplest components. We shall begin by defining *X* and *Y* as the simplest positive and negative matrices respectively and finding their general expressions.

Let $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$X^2 = X \bullet X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$X^3 = X^2 \bullet X = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 2+2 \\ 2+2 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$X^4 = X^3 \bullet X = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4+4 & 4+4 \\ 4+4 & 4+4 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

From these experimentations with the matrix *X*, we notice a clear pattern. In each consecutive matrix, the values of all four elements are twice as great as the values within the previous matrix. Therefore, as the power of *X* is increased by one, the values of all the elements within the matrix are multiplied by two. This trend is understandable since the process of matrix multiplication is row by column. Since there are two rows and columns in each column, the sum of the products of the first elements and the second elements is two times the original value.

Let $Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$Y^2 = Y \bullet Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$Y^3 = Y^2 \bullet Y = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & -2-2 \\ -2-2 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$Y^4 = Y^3 \bullet Y = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4+4 & -4-4 \\ -4-4 & 4+4 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

In each consecutive matrix, the values of all four elements are twice as great as the values within the previous matrix. Therefore, as the power of X is increased by one, the values of all the elements within the matrix are multiplied by 2. It should be noted that all of the elements remained positive.

Table 1: Matrices for X^n and Y^n

n	X^n	Y^n
5	$\begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}$	$\begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}$
6	$\begin{bmatrix} 32 & 32 \\ 32 & 32 \end{bmatrix}$	$\begin{bmatrix} 32 & -32 \\ -32 & 32 \end{bmatrix}$
7	$\begin{bmatrix} 64 & 64 \\ 64 & 64 \end{bmatrix}$	$\begin{bmatrix} 64 & -64 \\ -64 & 64 \end{bmatrix}$
8	$\begin{bmatrix} 128 & 128 \\ 128 & 128 \end{bmatrix}$	$\begin{bmatrix} 128 & -128 \\ -128 & 128 \end{bmatrix}$
9	$\begin{bmatrix} 256 & 256 \\ 256 & 256 \end{bmatrix}$	$\begin{bmatrix} 256 & -256 \\ -256 & 256 \end{bmatrix}$
10	$\begin{bmatrix} 512 & 512 \\ 512 & 512 \end{bmatrix}$	$\begin{bmatrix} 512 & -512 \\ -512 & 512 \end{bmatrix}$

Now that we have an idea of different patterns when X and Y are raised to an exponent ranging from 1 to 4, we can create a table for different values of X^n and Y^n with higher exponents. Using the graphing calculator, X and Y were entered through the Edit Matrix window. The expressions for X^n and Y^n were found by entering [X] or [Y}, ^ and n on the home screen. The different matrices are shown to the left in Table #1.

The general expression for X^n seems to be $\begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$. This can be derived from the fact that X doubles every time n increases. Since $X^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the starting power of 2 needs to be one less.

For Y^n, it seems to be $\begin{bmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{bmatrix}$. The same rules apply to Y as they do to X. The key difference is that two elements within the Y matrices maintain their negative signs. Having determined the basic general equations for both X and Y, we can combine X and Y into our simplest matrix binomial and attempt to find some interesting properties.

$$X + Y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(X+Y)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4+0 & 0+0 \\ 0+0 & 4+0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(X+Y)^3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 8+0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$(X+Y)^4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^3\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 16+0 & 0+0 \\ 0+0 & 16+0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

Once again, the pattern seems to be that each element in every consecutive matrix doubled. Of course, when the initial element is zero, all the elements of the same row and column equal zero.

However, the most important property that we notice is that $(X+Y)^n = X^n + Y^n$. We can see that this property holds true for all four trials done so far. The next step would be to test more values of n to ensure the property is true.

Table 2: Matrices of (X+Y)^n

n	$(X+Y)^n$
5	$\begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$
6	$\begin{bmatrix} 64 & 0 \\ 0 & 64 \end{bmatrix}$
7	$\begin{bmatrix} 128 & 0 \\ 0 & 128 \end{bmatrix}$
8	$\begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$
9	$\begin{bmatrix} 512 & 0 \\ 0 & 512 \end{bmatrix}$
10	$\begin{bmatrix} 1024 & 0 \\ 0 & 1024 \end{bmatrix}$

To examine several more powers of (X+Y)^n greater than 4, we will create a table displaying the various matrices corresponding to each different power. The same process used in determining the previous table is used except that the default table entered was $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The results are shown in Table #2.

The general statement for (X+Y)^n seems to be (X+Y)^n $=\begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}$.

This statement can be easily verified in all ten of the trials done for (X+Y)^n. When (X+Y) is examined carefully, the equation makes perfect sense because each time (X+Y) is multiplied to an exponent matrix of itself, the elements within the matrix are multiplied by 2. Therefore, as the power of (X+Y)^n increases, the value of each element within the matrix is also multiplied by 2.
So far, from the ten different numbers we have tested for n, we notice that (X+Y)^n = X^n + Y^n. This property seems to be correct for the positive and negative component matrices that we have selected and for the ten trials we have done. The next step would be to verify this property further with different positive and negative component matrices.

Having explored two very basic 2 by 2 matrices that contained elements with values of 1 or -1 and found the very interesting property of (X+Y)^n = X^n + Y^n, the next step would be explore more complex 2 by 2 matrices that are multiplied by a scalar multiple.
To investigate the positive and negative matrices, we shall suppose a matrix A where A=aX and a matrix B where B=bX and then let a = 2 and b =-3.

$$A = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4+4 & 4+4 \\ 4+4 & 4+4 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 16+16 & 16+16 \\ 16+16 & 16+16 \end{bmatrix} = \begin{bmatrix} 32 & 32 \\ 32 & 32 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 32 & 32 \\ 32 & 32 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 64+64 & 64+64 \\ 64+64 & 64+64 \end{bmatrix} = \begin{bmatrix} 128 & 128 \\ 128 & 128 \end{bmatrix}$$

Clearly, the values of the elements in each consecutive matrix are four times greater than the previous matrix. Therefore, with every increase in the power of A, the values of the elements within the product matrix are multiplied by a factor of 4. Once again, it should be noted that all elements stayed positive.

$$B = -3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 9+9 & -9-9 \\ -9-9 & 9+9 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} -54-54 & 54+54 \\ 54+54 & -54-54 \end{bmatrix} = \begin{bmatrix} -108 & 108 \\ 108 & -108 \end{bmatrix}$$

$$B^4 = \begin{bmatrix} -108 & 108 \\ 108 & -108 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 324+324 & -324-324 \\ -324-324 & 324+324 \end{bmatrix} = \begin{bmatrix} 648 & -648 \\ -648 & 648 \end{bmatrix}$$

The value of each element in the subsequent matrix is now multiplied by a power of -6. In other words, every time the power of *B* is increased by 1, the values of every element are multiplied by a factor of -6. It should be noted that multiplication by a factor of -6 does change the value of the elements from positive to negative and negative to positive.

Table 3: Matrices for A^n and B^n

n	A^n	B^n
5	$\begin{bmatrix} 512 & 512 \\ 512 & 512 \end{bmatrix}$	$\begin{bmatrix} -3888 & 3888 \\ 3888 & -3888 \end{bmatrix}$
6	$\begin{bmatrix} 2048 & 2048 \\ 2048 & 2048 \end{bmatrix}$	$\begin{bmatrix} 23328 & -23328 \\ -23328 & 23328 \end{bmatrix}$
7	$\begin{bmatrix} 8192 & 8192 \\ 8192 & 8192 \end{bmatrix}$	$\begin{bmatrix} -139968 & 139968 \\ 139968 & -139968 \end{bmatrix}$
8	$\begin{bmatrix} 8192 & 8192 \\ 8192 & 8192 \end{bmatrix}$	$\begin{bmatrix} -139968 & 139968 \\ 139968 & -139968 \end{bmatrix}$

Once again, we have found the general patterns for matrices *A* and *B* when they are raised to a power under 5. For higher powers, the matrices will be found with the graphing calculator using the same process described earlier but with a different initial matrix. The results are shown in Table #3.
Based on Table #3 and previous observations, we can find the general expression for A^n and B^n.
For A^n, the general expression seems to be $A^n = \begin{bmatrix} 2(4^{n-1}) & 2(4^{n-1}) \\ 2(4^{n-1}) & 2(4^{n-1}) \end{bmatrix}$. This formula was derived from the observation that the initial values of A^n are all 2. Therefore, the two outside of the brackets determine the initial value of this geometric sequence. Afterwards, the values of each subsequent matrix is multiplied by a factor of 4, so within the brackets, we have 4^{n-1}.

Then for B^n, we find the general expression to be $B^n = B^n = \begin{bmatrix} -3((-6)^{n-1}) & 3((-6)^{n-1}) \\ 3((-6)^{n-1}) & -3((-6)^{n-1}) \end{bmatrix}$.

This formula was derived the same way as the previous one except that the initial values of the first matrix were -3 and 3 and the values of each subsequent matrix were multiplied by a factor of -6.
After looking at the two different general expression and noticing that if we factor out the values of *a* and *b* from the general expression of A^n and B^n respectively, we can come up with a general expression for A^n and B^n no matter what scalar value was multiplied. For *A*, the general form seems to be $A^n = \begin{bmatrix} a^n 2^{n-1} & a^n 2^{n-1} \\ a^n 2^{n-1} & a^n 2^{n-1} \end{bmatrix}$ where *a* is the scalar multiplied to *X*.

For *B*, it seems to be $B^n = \begin{bmatrix} b^n 2^{n-1} & -b^n 2^{n-1} \\ -b^n 2^{n-1} & b^n 2^{n-1} \end{bmatrix}$, b is the scalar multiplied to *Y*.

These two new formulas give another interesting property. Using deductive reasoning skills, the general formulas for both *A* and *B* seems to be the original formulas of X to the power of *n* multiplied by the scalar to the power of *n* as well.

This can be expressed as $(a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})^n = a^n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^n = a^n \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}^n = \begin{bmatrix} a^n 2^{n-1} & a^n 2^{n-1} \\ a^n 2^{n-1} & a^n 2^{n-1} \end{bmatrix}$

and $(b\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix})^n = b^n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^n = b^n \begin{bmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{bmatrix}^n = \begin{bmatrix} b^n 2^{n-1} & b^n 2^{n-1} \\ -b^n 2^{n-1} & b^n 2^{n-1} \end{bmatrix}$.

Therefore, the new property should be summarized as $(aX)^n = a^n X^n$ and $(bY)^n = b^n Y^n$.

The next step now would be test the same property of $(X+Y)^n = X^n + Y^n$ when the values of *X* and *Y* have been multiplied by a scalar multiple.

$$(A+B) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 1+25 & -5-5 \\ -5-5 & 1+25 \end{bmatrix} = \begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix}$$

$$(A+B)^3 = \begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} -26-50 & 130+10 \\ 130+10 & -26-50 \end{bmatrix} = \begin{bmatrix} -76 & 140 \\ 140 & -76 \end{bmatrix}$$

$$(A+B)^4 = \begin{bmatrix} -76 & 140 \\ 140 & -76 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 76+700 & -380-140 \\ -380-140 & 76+700 \end{bmatrix} = \begin{bmatrix} 776 & -520 \\ -520 & 776 \end{bmatrix}$$

$(A+B)^n$ seems to follow a much more complicated pattern. First, it should be noted that the signs of each corresponding element changes with every consecutive matrix. Second, simple analysis of the different numbers fails to yield a straight forward formula. Finally, when the pattern is examined closely, it appears that the value of $(A+B)^n$ is equal to the values of $A^n + B^n$. For example, $(A+B)^2$ is clearly equal to $A^2 + B^2$. Once again the property seems to hold true.

Once again, it is time to put all of the exponents of $(A+B)^n$ into a table. Using the same process as described before but with a different initial matrix, the results were founded and then shown in Table #3.

First, it must be noted that the property $(X+Y)^n = X^n + Y^n$ still holds try for the next four powers of n that we have tested. Since the property holds true for one scalar multiple of the simplest form, the next step would be to see if it works with all scalar multiples.

Second, based simply previous calculations and on Table #3, it is actually almost impossible to find an equation because the pattern is extremely complex.

Table 4: Matrices of $(A+B)^n$

n	$(A+B)^n$
5	$\begin{bmatrix} -3376 & 4400 \\ 4400 & -3376 \end{bmatrix}$
6	$\begin{bmatrix} 25376 & -21280 \\ -21280 & 25376 \end{bmatrix}$
7	$\begin{bmatrix} -131776 & 148160 \\ 148160 & -131776 \end{bmatrix}$
8	$\begin{bmatrix} 872576 & -807040 \\ -807040 & 872576 \end{bmatrix}$

Third, however, using the property of $(X+Y)^n = X^n + Y^n$, the general expression can be determined. Quite interestingly, the simplest expression that can be found is just the equation of A^n and B^n combined. Therefore, the general expression for $(A+B)^n$ is

$$(A+B)^n = \begin{bmatrix} 2(4^{n-1}) - 3((-6)^{n-1}) & 2(4^{n-1}) + 3((-6)^{n-1}) \\ 2(4^{n-1}) + 3((-6)^{n-1}) & 2(4^{n-1}) - 3((-6)^{n-1}) \end{bmatrix}$$

Using the general expressions arrived for any scalar multiple of X and Y.

$$(A+B)^n = \begin{bmatrix} a^n 2^{n-1} + b^n 2^{n-1} & a^n 2^{n-1} - b^n 2^{n-1} \\ a^n 2^{n-1} - b^n 2^{n-1} & a^n 2^{n-1} + b^n 2^{n-1} \end{bmatrix}.$$

These two equations further support the first property that we found: $(X+Y)^n = X^n + Y^n$.

Finally, we can input simple variables a and b to stand for our scalar multiples to show that $(X+Y)^n = X^n + Y^n$ is true all values of a and b. We shall let $M = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$, since M is matrix binomial and should be equal to $(X+Y)$. Our first step shall be to test the first 3 powers of M and $(A+B)$ to see if the property hold true.

$$A + B = \begin{bmatrix} a & a \\ a & a \end{bmatrix} + \begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$M = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$\therefore M = A + B$$

$$A^2 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} a^2 + a^2 & a^2 + a^2 \\ a^2 + a^2 & a^2 + a^2 \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} b+b & -b-b \\ -b-b & b+b \end{bmatrix} = \begin{bmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix} + \begin{bmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{bmatrix} = \begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + 2db + b^2 + a^2 - 2db + b^2 & a^2 + 2db + b^2 + a^2 - 2db - b^2 \\ a^2 + 2db + b^2 + a^2 - 2db - b^2 & a^2 + 2db + b^2 + a^2 - 2db + b^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix}$$

$$\therefore M^2 = A^2 + B^2$$

$$A^3 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 2a^3+2a^3 & 2a^3+2a^3 \\ 2a^3+2a^3 & 2a^3+2a^3 \end{bmatrix} = \begin{bmatrix} 4a^3 & 4a^3 \\ 4a^3 & 4a^3 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{bmatrix}\begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} 2b^3+2b^3 & -2b^3-2b^3 \\ -2b^3-2b^3 & 2b^3+2b^3 \end{bmatrix} = \begin{bmatrix} 4b^3 & -4b^3 \\ -4b^3 & 4b^3 \end{bmatrix}$$

$$A^3 + B^3 = \begin{bmatrix} 4a^3 & 4a^3 \\ 4a^3 & 4a^3 \end{bmatrix} + \begin{bmatrix} 4b^3 & -4b^3 \\ -4b^3 & 4b^3 \end{bmatrix} = \begin{bmatrix} 4a^3+4b^3 & 4a^3-4b^3 \\ 4a^3-4b^3 & 4a^3+4b^3 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{bmatrix}\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$= \begin{bmatrix} 4a^3+2a^2b+2ab^2+4b^3-2a^2b-2ab^2 & 4a^3+2a^2b+2ab^2-4b^3-2a^2b-2ab^2 \\ 4a^3+2a^2b+2ab^2-4b^3-2a^2b-2ab^2 & 4a^3+2a^2b+2ab^2+4b^3-2a^2b-2ab^2 \end{bmatrix}$$

$$= \begin{bmatrix} 4a^3+4b^3 & 4a^3-4b^3 \\ 4a^3-4b^3 & 4a^3+4b^3 \end{bmatrix}$$

$$\therefore M^3 = A^3 + B^3$$

Clearly, the property holds true for the first three positive powers: 1, 2 and 3. Therefore, the general statement that $M^n = A^n + B^n$ seems to hold true and in turn, the general property discovered earlier, $(X+Y)^n = X^n + Y^n$, also holds true.

The other property that we had discovered earlier was $(aX)^n = a^n X^n$ and $(bY)^n = b^n Y^n$. Using this property, we simplify $A^n = (aX)^n = a^n X^n$ and $B^n = (bY)^n = b^n Y^n$. At last, we can substitute and get a general equation for M to be $M = a^n X^n + b^n Y^n$. Nevertheless,, this formula seems to be of little use right now.

Of course, the earlier formula that we found for $(A+B)^n$ still applies. After applying it to M, the simplest general expression for M is $M^n = \begin{bmatrix} a^n 2^{n-1} + b^n 2^{n-1} & a^n 2^{n-1} - b^n 2^{n-1} \\ a^n 2^{n-1} - b^n 2^{n-1} & a^n 2^{n-1} + b^n 2^{n-1} \end{bmatrix}$.

Having identified a certain general expression for M, the final step of this long process of investigating matrix binomials would be testing if this general expression works for any number chosen for *a*, *b*, and *n*.

To calculate the result of the matrix, the values of *a, b* and *n* are substituted into the equation $M^n = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}^n$. Then the resulting matrix without the exponent *n* is inputted into the graphing calculator through the Matrix Edit screen and finally, the product matrix is found by raising the inputted matrix to the power of *n* in the home screen. The general expression $M^n = \begin{bmatrix} a^n 2^{n-1} + b^n 2^{n-1} & a^n 2^{n-1} - b^n 2^{n-1} \\ a^n 2^{n-1} - b^n 2^{n-1} & a^n 2^{n-1} + b^n 2^{n-1} \end{bmatrix}$ is used to predict the matrix. By substituting the three variables into the expression, M^n can be found and the two values can be compared.

Table 5: Testing the Validity of the General Statement

a	*b*	*n*	Matrix Calculated (M^n=)	Matrix Predicted (M^n=)
3	4	3	$=\begin{bmatrix}3+4 & 3-4\\ 3-4 & 3+4\end{bmatrix}^3=\begin{bmatrix}7 & -1\\ -1 & 7\end{bmatrix}^3$ $=\begin{bmatrix}364 & -148\\ -148 & 364\end{bmatrix}$	$=\begin{bmatrix}3^3 2^2+4^3 2^2 & 3^3 2^2-4^3 2^2\\ 3^3 2^2-4^3 2^2 & 3^3 2^2+4^3 2^2\end{bmatrix}$ $=\begin{bmatrix}364 & -148\\ -148 & 364\end{bmatrix}$
7	-2	5	$=\begin{bmatrix}7-2 & 7+2\\ 7+2 & 7-2\end{bmatrix}^5=\begin{bmatrix}5 & 9\\ 9 & 5\end{bmatrix}^5$ $=\begin{bmatrix}268400 & 269424\\ 269424 & 268400\end{bmatrix}$	$=\begin{bmatrix}7^5 2^4-2^5 2^4 & 7^5 2^4+2^5 2^4\\ 7^5 2^4+2^5 2^4 & 7^5 2^4-2^5 2^4\end{bmatrix}$ $=\begin{bmatrix}268400 & 269424\\ 269424 & 268400\end{bmatrix}$
-5	3	4	$=\begin{bmatrix}-5+3 & -5-3\\ -5-3 & -5+3\end{bmatrix}^4=\begin{bmatrix}-2 & -8\\ -8 & -2\end{bmatrix}^4$ $=\begin{bmatrix}5648 & 4352\\ 4352 & 5648\end{bmatrix}$	$=\begin{bmatrix}-5^4 2^3+3^4 2^3 & -5^4 2^3-3^4 2^3\\ -5^4 2^3-3^4 2^3 & -5^4 2^3+3^4 2^3\end{bmatrix}^3$ $=\begin{bmatrix}5648 & 4352\\ 4352 & 5648\end{bmatrix}$
-3	6	7	$=\begin{bmatrix}-3+6 & -3-6\\ -3-6 & -3+6\end{bmatrix}^7=\begin{bmatrix}3 & -9\\ -9 & 3\end{bmatrix}^7$ $=\begin{bmatrix}17775936 & -18055872\\ -18055872 & 17775936\end{bmatrix}$	$=\begin{bmatrix}-3^7 2^6+6^7 2^6 & -3^7 2^6-6^7 2^6\\ -3^7 2^6-6^7 2^6 & -3^7 2^6+6^7 2^6\end{bmatrix}$ $=\begin{bmatrix}17775936 & -18055872\\ -18055872 & 17775936\end{bmatrix}$
12	2	4	$=\begin{bmatrix}12+2 & 12-2\\ 12-2 & 12+2\end{bmatrix}^4=\begin{bmatrix}14 & 10\\ 10 & 14\end{bmatrix}^4$ $=\begin{bmatrix}166016 & 165760\\ 165760 & 166016\end{bmatrix}$	$=\begin{bmatrix}12^4 2^3+2^4 2^3 & 12^4 2^3-2^4 2^3\\ 12^4 2^3-2^4 2^3 & 12^4 2^3+2^4 2^3\end{bmatrix}$ $=\begin{bmatrix}166016 & 165760\\ 165760 & 166016\end{bmatrix}$
2	-7	2	$=\begin{bmatrix}2-7 & 2+7\\ 2+7 & 2-7\end{bmatrix}^2=\begin{bmatrix}-5 & 9\\ 9 & -5\end{bmatrix}^2$ $=\begin{bmatrix}106 & -90\\ -90 & 106\end{bmatrix}$	$=\begin{bmatrix}2^2 2^1+7^2 2^1 & 2^2 2^1-7^2 2^1\\ 2^2 2^1-7^2 2^1 & 2^2 2^1+7^2 2^1\end{bmatrix}$ $=\begin{bmatrix}106 & -90\\ -90 & 106\end{bmatrix}$
-6	1	1	$=\begin{bmatrix}-6+1 & -6-1\\ -6-1 & -6+1\end{bmatrix}^2=\begin{bmatrix}-5 & -7\\ -7 & -5\end{bmatrix}^2$ $=\begin{bmatrix}-5 & -7\\ -7 & -5\end{bmatrix}$	$=\begin{bmatrix}-6^1 2^0+1^1 2^0 & -6^1 2^0-1^1 2^0\\ -6^1 2^0-1^1 2^0 & -6^1 2^0+1^1 2^0\end{bmatrix}$ $=\begin{bmatrix}-5 & -7\\ -7 & -5\end{bmatrix}$

For all six randomly selected values of *a, b,* and *n,* the calculated matrix and the predicted matrix was the same, therefore, the general expression continues to hold true for every test that has been done. Since this general expression was just an expanded form of the expression $M^n = A^n + B^n$, this property also continues to hold true.

Having done numerous tests, the next step would be proving the property $M^n = A^n + B^n$ and general statement $M^n = \begin{bmatrix} a^n 2^{n-1} + b^n 2^{n-1} & a^n 2^{n-1} - b^n 2^{n-1} \\ a^n 2^{n-1} - b^n 2^{n-1} & a^n 2^{n-1} + b^n 2^{n-1} \end{bmatrix}$ inductively. One possible method to use would be mathematical induction. However, the formal proof will not be covered in this project.

Concluding Remarks

The entire project was built around investigating and proving different properties of matrix binomials and their components. One of the most important properties of matrix binomials that was discovered was the property $M^n = A^n + B^n$ which can also be written as $(A+B)^n = A^n + B^n$. This important property allows us to factor out matrix binomials and thus allows us to manipulate matrix binomials much more easily.

The general statement $M^n = \begin{bmatrix} a^n 2^{n-1} + b^n 2^{n-1} & a^n 2^{n-1} - b^n 2^{n-1} \\ a^n 2^{n-1} - b^n 2^{n-1} & a^n 2^{n-1} + b^n 2^{n-1} \end{bmatrix}$ was also found through this project. Although a large formula, this general statement can help us quickly analyze and factor matrix binomials. Using this formula, we can factor out the base matrices X and Y rapidly in order to find the scalar multiples that were multiplied to them. Nevertheless, despite the fact that many tests were done to test these properties and the general statement, they have yet to be proven true for all possible cases. Some cases that were missed include non-integer values for a, b, and n.

Another interesting property that was found was $(aX)^n = a^n X^n$. This property could probably be extended to all matrices and not just matrix binomials, but insufficient time was spent investigating this property. More tests on different numbers and different base matrices must be done to verify this property further.

Overall, the research was a quick glimpse into a major property ($M^n = A^n + B^n$) of a matrix binomial. Many different tests were done to verify and to try to find a counter-example, but so far, all tests support this property. Further tests and an inductive proof is needed to verify this property. Of course, a general statement was also found based upon this property. The accuracy of this general statement is dependent on the accuracy of this property.