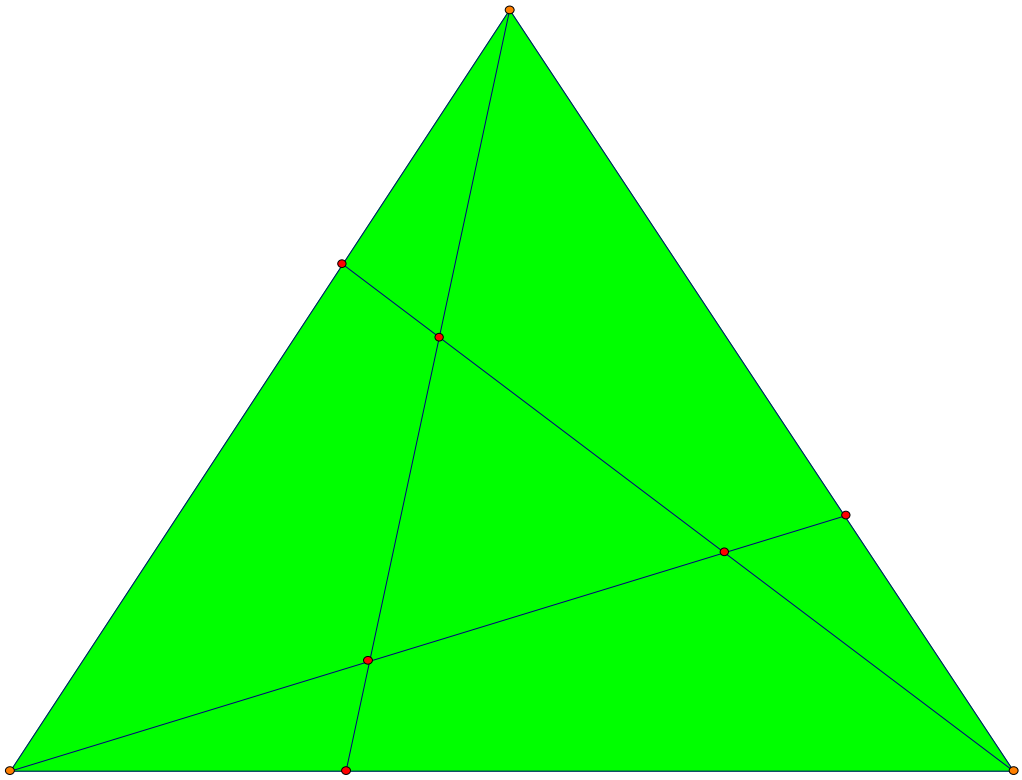
Math IA- Type 1
The Segments of a Polygon

Abhinav Jain
IB Higher Level Assignment: Internal Assessment
Type 1: Modeling
Mr. Murgatroyd
Date: 15/03/2009
Word Count:

1. In an equilateral triangle ABC, a line segment is drawn from each vertex to a point on the opposite side so that the segment divides the side in the ratio 1:2, creating another equilateral triangle DEF.

a)
i)



ii)
Measurements and drawing shown above has been made through the Geometer's Sketchpad package.
Measure of one side of the ΔABC = 12cm
Measure of one side of the ΔDEF = 5cm
iii)
The areas have also been calculated using the Geometer's Sketchpad package and are show in the diagram above.

For ΔABC = 62.5cm^2

For ΔDEF = 8.8 cm^2

In order to find the ratio, one needs to divide the area of ΔABC by the area of the ΔDEF which will give one the ratio of the area of the bigger triangle to the smaller triangle.

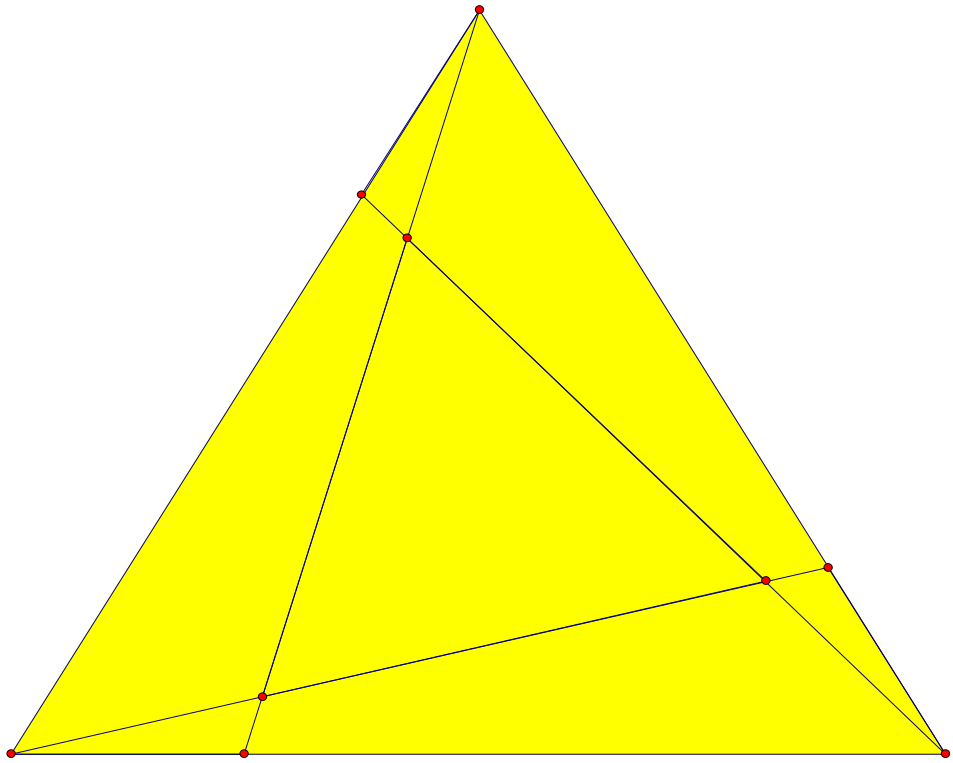
Therefore

62.5cm^2 ÷ 8.8 cm^2 = 7:1

The ratio between the areas of the equilateral ΔABC to ΔDEF when the segment divides the side in the ratio 1:2 is 7:1.

b) In order to repeat the procedure above for at least two other side ratios, 1: n The two ratios chosen are 1:3 and 1:4 for no specific reason.

Ratio of Sides = 1:3



Again, the diagram above and values were obtained and created using Geometer's Sketchpad Package.

For ΔABC = 62.0 cm^2

For ΔDEF = 19.1 cm^2

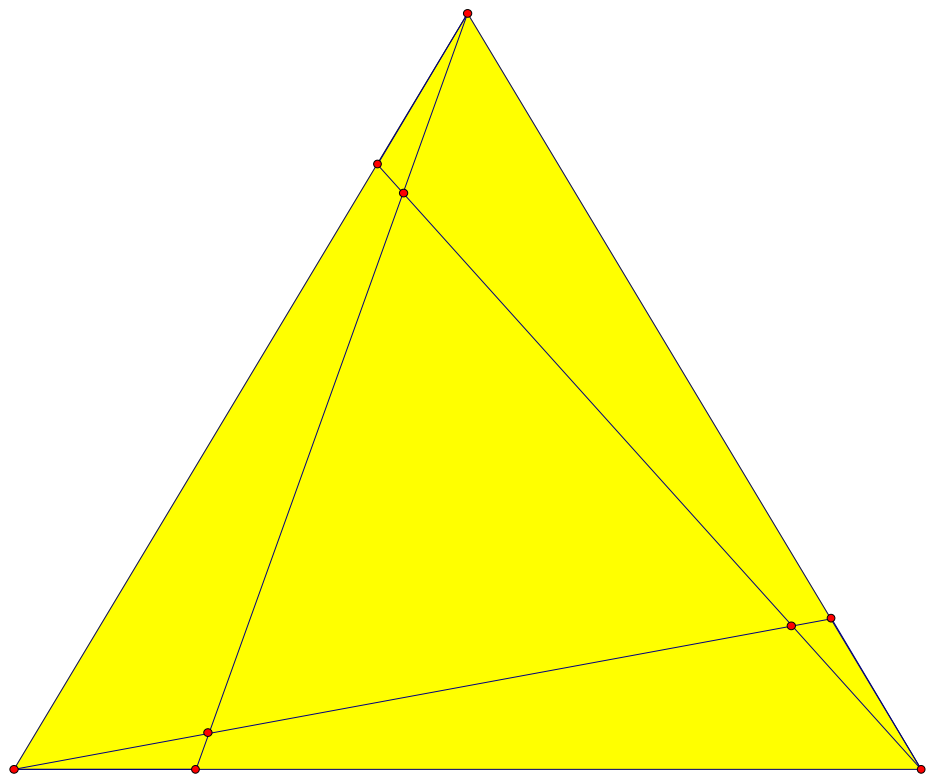
In order to find the ratio, one needs to divide the area of ΔABC by the area of the ΔDEF which will give one the ratio of the area of the bigger triangle to the smaller triangle.

Therefore

62.0 cm^2/19.1cm^2 = 13:4

The ratio between the areas of the equilateral ΔABC to ΔDEF when the segment divides the side in the ratio 1:3 is 13:4.

Ratio of Sides = 1:4



For ΔABC = 62.1 cm^2
For ΔDEF = 26.6 cm^2

In order to find the ratio, one needs to divide the area of ΔABC by the area of the ΔDEF which will give one the ratio of the area of the bigger triangle to the smaller triangle.

Therefore

62.1 cm^2/26.6 cm^2 = 7:3

The ratio between the areas of the equilateral ΔABC to ΔDEF when the segment divides the side in the ratio 1:4 is 7:3.

c) The following table compares the values of the ratios of the sides and the ratios of the areas of the triangles so that one can deduce a relationship by analyzing the results below.

Ratios of sides of triangles = 1:n	Ratios of areas of equilateral triangles(Ratio of bigger triangle to smaller triangle)
1:2	7:1
1:3	13:4
1:4	7:3

If one looks at the values above, it is hard to be able to determine a relationship as the values of the ratio of areas are not very similar at all. However if one were to change the ratio of areas for the ratio of sides 1:4, from 7:3 to 21:9, then it would look like the following.

Ratios of sides of triangles = 1:n	Ratios of areas of equilateral triangles(Ratio of bigger triangle to smaller triangle)
1:2	7:1
1:3	13:4
1:4	21:9

Now if one were to look at the values of ratios above, then one can see relationships both in the numerator and denominator of the ratios. One can see that the denominators are actually squares of integers such as 1, 2 and 3. The following table helps to analyze the situation better.

Ratios of sides of triangles = 1:n	Denominator values of the area ratio of equilateral triangles
1:2	1^2
1:3	2^2
1:4	3^2

Looking over to the left of the table, one can see that as n increases, the value which is squared by which the denominator is found also increases in the same proportion which is 1. However, one also sees that the value which is squared to form the denominator is actually one less than n which illustrates the ratio

produced by the segment that divides the ratio in its specific values. If one were to put this in mathematical terms, then it would be the following.

Value of denominator of ratio of areas of equilateral triangles = $(n - 1)^2$

This is because the value which is squared to form the denominator is one less than n and after it has been subtracted, the value is squared, hence, forming the equation above.

Now moving onto the numerator, the following table helps analyze the situation better.

Ratios of sides of triangles = 1:n	Numerator values of the area ratio of equilateral triangles
1:2	7
1:3	13
1:4	21

Looking at the values above, one can notice that the difference between numerator values is increasing. Not only that, it is increasing in a specific value which creates a pattern. The very first time when n is increased by 1, the value increases by 6. When n is again increased by 1, then the value increases by 8. A pattern that is noticed here is that for each and every n value, if it were to be squared and then n was added to that value again, it would only be 1 unit away from the numerator value. If one were to apply this into mathematical terms, then it would look like the following.

$n^2 + n + 1$ = Numerator values

Therefore, if the two equations that were conjectured above are put together, they would look like the following

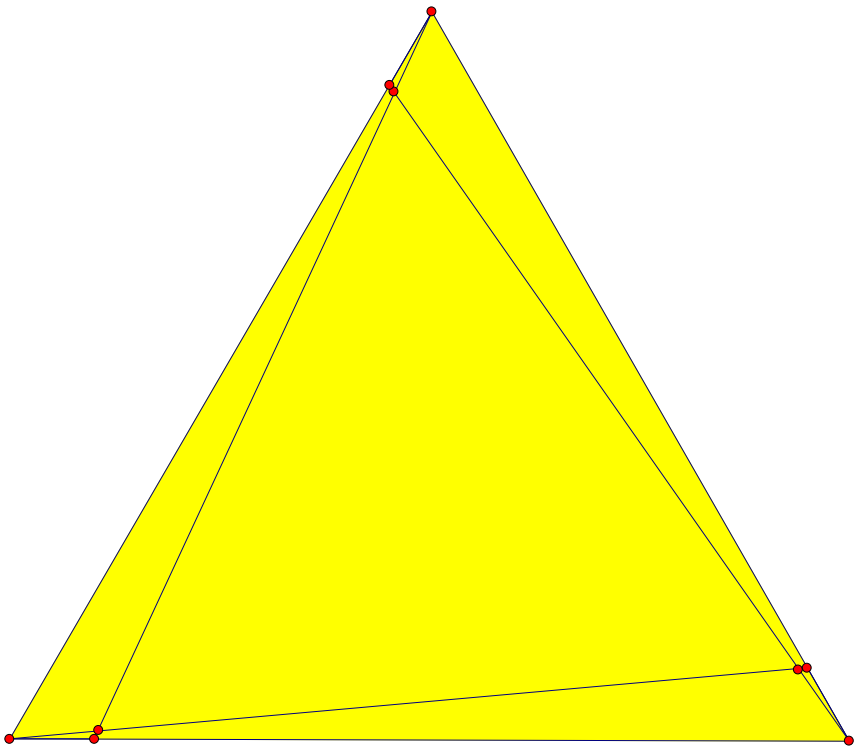
$n^2 + n + 1 \div (n - 1)^2$

d) Before this conjecture is proved analytically, it is very important to understand the basics of similarity. Triangles are similar if their corresponding angles are congruent and the ratio of their corresponding sides is in proportion. The name for this proportion is known as the similarity ratio.

In order to prove the conjecture above, one needs to find the area of the bigger triangle and divide it by the area of the smaller triangle. If the produced equation matches the equation above, then the conjecture has been proved analytically.

The conjecture has been proved on the following page.

In to order to able to test the validity of this conjecture, another triangle was produced using the geometer's sketch pad and the conjecture is validated if the ratio produced from GSP matches the value of the ratio provided by the conjecture.



The ratio the segment divides the side in is 1:9

From our conjecture which is

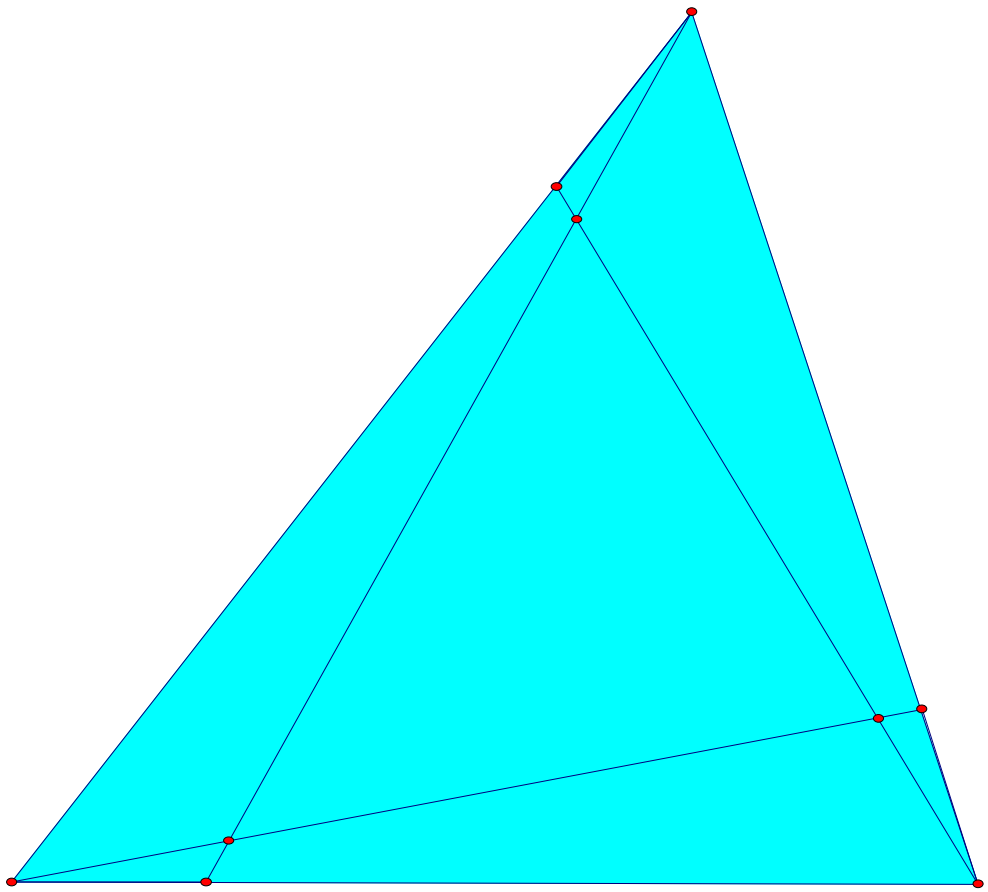
$(n^2 + n + 1) \div (n - 1)^2$

Therefore if 9 is substituted in place on n, then the following answer is retrieved.

$(9^2 + 9 + 1) \div (9-1)^2$
$= 1.42$

Therefore both the values from GSP and the conjecture match, proving the conjecture to be valid.

2) In order to be able to prove whether this conjecture is valid for non-equilateral triangles, another sketch is made using the GSP. However, this time a non equilateral triangle is made. If the value of the ratio of the area matches the ratio from the conjecture, then the equations holds true for non-equilateral triangles. The ratio of sides that will be used will 1:4



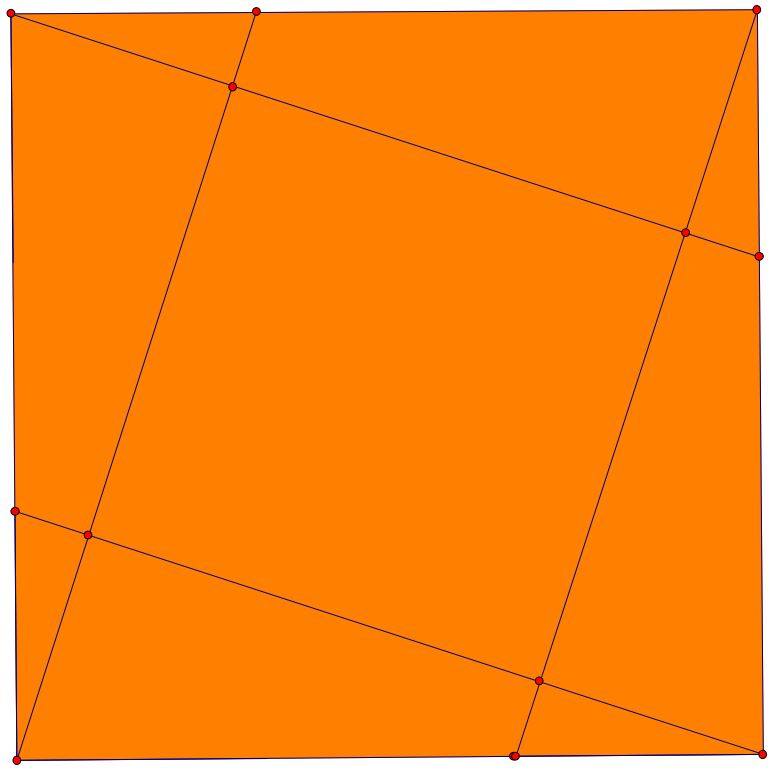
The ratio between the areas of the two triangles show above is 2.34

In order to prove whether the conjecture is valid for non-equilateral triangles, the value retrieved from the conjecture once 4 is substituted into the equation should equal somewhere around the value of 2.34.

$(n^2 + n + 1) \div (n - 1)^2$
$= (4^2 + 4 + 1) \div (4-1)^2$
$= 2.33$

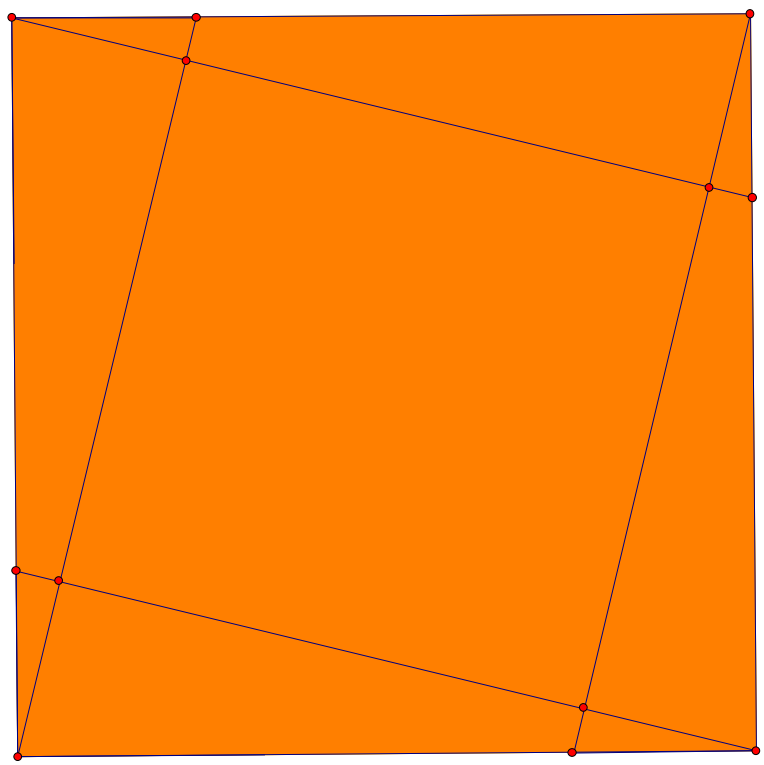
These two answers are very close to each other and taking the uncertainties of the GSP into account, this is an accurate answer. Therefore, the conjecture holds true for non- equilateral triangles. The limitations and uncertainties of GSP will be discussed at the end of this assignment.

3) a) In order to compare the area of the inner square to the area of the original square, GSP was used again to construct a square where the segment divided the ratio of the sides into 1:2. The square constructed is shown below



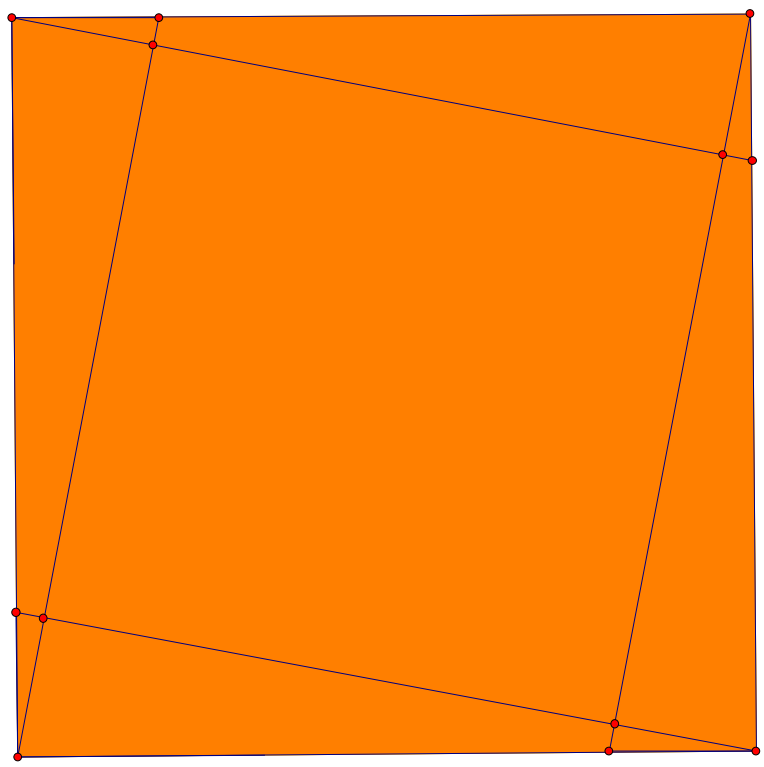
The area of the inner square compared to the original square is given above to be around 0.4 which is the same as 2:5 in ratio format.

b) In order to find how the areas compare if each side is divided in the ratio 1: n, two GSP sketches of squares need to be made. One ratio used is going to be 1:3 and 1:4. The one shown below is 1:3.



The ratio of the area of inner square to the area of the original square for the square above is believed to be about 9:17.

The sides of the square illustrated below have been divided into the ratio of 1:4.



The answer retrieved above is about 0.62 which is close to the ratio of 16:26.

Looking at the two squares and their ratios above, a pattern that seems to be developing can be seen.

The following table compares the values of the ratios of the sides and the ratios of the areas of the squares so that one can deduce a relationship by analyzing the results below.

Ratios of sides of squares = 1:n	Ratios of areas of squares(Ratio of

	smaller square to bigger square)
1:2	2:5
1:3	9:17
1:4	16:26

If one looks at the values above, it is hard to be able to determine a relationship as the values of the ratio of areas are not very similar at all. However if one were to change the ratio of areas for the ratio of sides 1:2, from 2:5 to 4:10, then it would look like the following.

Ratios of sides of squares = 1:n	Ratios of areas of squares(Ratio of smaller square to bigger square)
1:2	4:10
1:3	9:17
1:4	16:26

Now if one were to look at the values of ratios above, then one can see relationships both in the numerator and denominator of the ratios. One can see that the numerators are actually squares of integers such as 2, 3 and 4. The following table helps to analyze the situation better.

Ratios of sides of squares = 1:n	Numerator values of the area ratio of the squares
1:2	2^2
1:3	3^2
1:4	4^2

Therefore, the numerator is just the value of n squared. The following puts it into a mathematical equation.

$= n^2$

Now moving onto the denominator, the following table helps analyze the situation better.

Ratios of sides of squares = 1:n	Denominator values of the area ratio of squares
1:2	10
1:3	17
1:4	26

Looking at the values above, one can notice that the difference between denominator values is increasing. Not only that, it is increasing in a specific value which creates a pattern. The very first time when n is increased by 1, the value increases by 7. When n is again increased by 1, then the value increases by 9. A pattern that is noticed here is that for each and every n value, if it were to be squared and then twice the value of n was added to that value again, it would only be 2 units away from the denominator value. If one were to apply this into mathematical terms, then it would look like the following.

$n^2 + 2n + 2$ = Denominator values

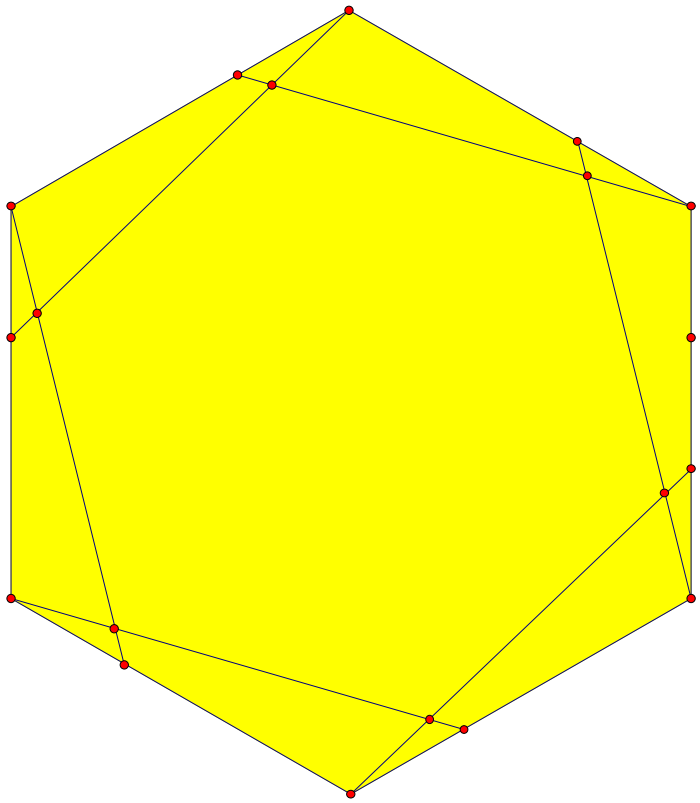
Therefore, if the two equations that were conjectured above are put together, they would look like the following

$n^2 \div n^2 + 2n + 2$

c) The conjecture for the ratios of the areas of the square is proved on the following page.

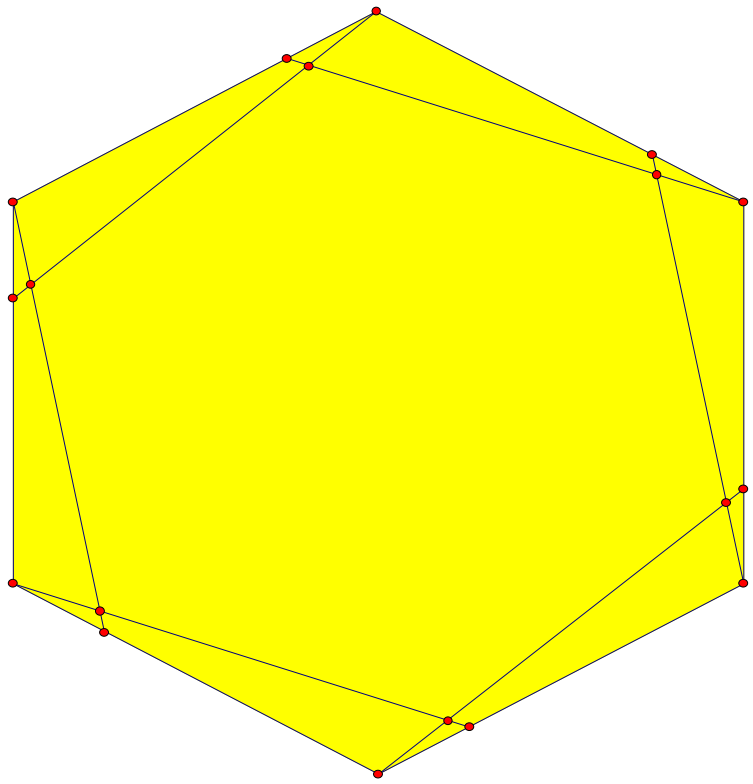
4) In order to investigate the relationship in another regular polygon, a hexagon has been chosen. In order to find the relationship between its original to its inner area, three sketches of hexagons need to be performed in the ratio of the sides being 1:2, 1:3 and 1:4.

The ratio of the sides being 1:2 is illustrated below.



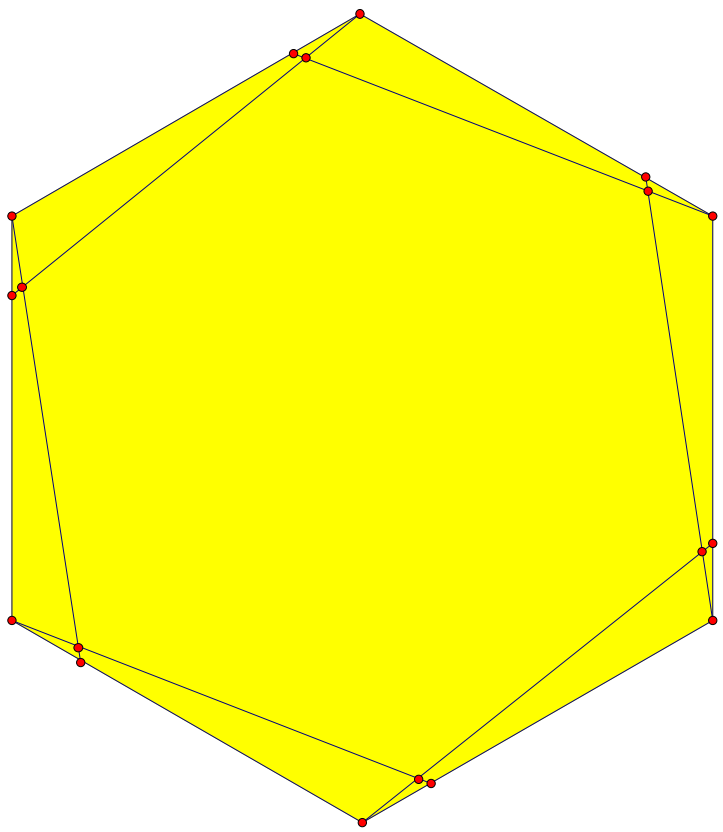
The answer retrieved above is 1.44 which when put into a fraction gives a ratio of the bigger hexagon to the inner hexagon being 13:9.

The ratio of the sides being 1:3 is illustrated below.



The answer retrieved above is 1.31 which when put into a fraction gives a ratio of the bigger hexagon to the inner hexagon being 21:16.

The ratio of the sides being 1:4 is illustrated below.



The answer retrieved above is 1.23 which when put into a fraction gives a ratio of the bigger hexagon to the inner hexagon being 31:25.

The following table compares the values of the ratios of the sides and the ratios of the areas of the hexagons so that one can deduce a relationship by analyzing the results below.

Ratios of sides of hexagons = 1:n	Ratios of areas of hexagons(Ratio of bigger hexagon to smaller hexagon)
1:2	13:9
1:3	21:16
1:4	31:25

Now if one were to look at the values of ratios above, then one can see relationships both in the numerator and denominator of the ratios. One can see that the denominators are actually squares of integers such as 3, 4 and 5. The following table helps to analyze the situation better.

Ratios of sides of hexagons = 1:n	Denominator values of the area ratio of hexagons
1:2	3^2
1:3	4^2
1:4	5^2

Looking over to the left of the table, one can see that as n increases, the value which is squared by which the denominator is found also increases in the same proportion which is 1. However, one also sees that the value which is squared to form the denominator is actually one more than n which illustrates the ratio produced by the segment that divides the ratio in its specific values. If one were to put this in mathematical terms, then it would be the following.

Value of denominator of ratio of areas of equilateral triangles = $(n + 1)^2$

This is because the value which is squared to form the denominator is one less than n and after it has been subtracted, the value is squared, hence, forming the equation above.

Now moving onto the numerator, the following table helps analyze the situation better.

Ratios of sides of hexagons= 1:n	Numerator values of the area ratio of hexagons
1:2	13
1:3	21
1:4	31

Looking at the values above, one can notice that the difference between numerator values is increasing. Not only that, it is increasing in a specific value which creates a pattern. The very first time when n is increased by 1, the value increases by 8. When n is again increased by 1, then the value increases by 10. A pattern that is noticed here is that for each and every n value, if it were to be squared and then three times n was added to that value again, it would only be 3 units away from the numerator value. If one were to apply this into mathematical terms, then it would look like the following.

$n^2 + 3n + 3$ = Numerator values

Therefore, if the two equations that were conjectured above are put together, they would look like the following

$n^2 + 3n + 3 \div (n + 1)^2$

This conjecture is proved analytically on the next page.

If the segments were constructed in a similar manner in other regular polygons, would a similar relationship exist?

I believe so that a similar relationship would exist because of the conjectures found of the different shapes. The table below shows the conjectures that were found for the three different shapes that were investigated in this assignment above.

Shape	Conjectures in terms of n
Triangle	$(n^2 + n + 1) \div (n - 1)^2$
Square	$n^2 \div n^2 + 2n + 2$
Hexagon	$n^2 + 3n + 3 \div (n + 1)^2$

The relationship that existed between all three was the fact that both the inner and outer shapes lengths were all found through the means of similar triangles. Around the inner shape existed small and big triangles which were similar dude to the Angle Angle similarity that was shared between the triangles. Through similarity, the lengths of the sides could be found which could be subtracted from the whole length of the outer shape to give the inner shapes length. Once this was calculated, the shape was substituted into the area equation to give the conjecture. So, in these terms, the shapes shared a relationship due to the Angle Angle similarity. Another relationship was the fact that the denominator or

the numerator included a quadratic equation. It was there to represent the bigger shape in all of the cases. For example, the quadratic equation either represented the outer and the bigger triangle, square and the hexagon. Therefore, in other polygons, one can predict that this will also be the case. Another relationship would be the fact that the smaller shape were represented by something that included n^2. Therefore, one can also deduce that the other smaller shapes such as of a pentagon or an octagon will also follow this rule.

Limitations of this assignment:

There was one main limitation that was related to the technological aspect of this assignment. The main problem of this assignment was using GSP itself. One can say that the values calculated by GSP for the different areas and lengths were not as accurate as they could possibly have been. This is partly due to human error as while we try to use the mouse to change the different lengths, it is nearly impossible to be able to move the hand with such precision so that every one of our lengths is consistent with each other therefore resulting in small inaccuracies caused due to the way GSP was programmed and the mouse that was in use while making the different shapes.