MATH 20IB SL PORTFOLIO TYPE 1
STELLAR NUMBERS

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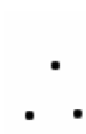
Introduction:

In this portfolio, I will investigate a pattern of numbers that can be represented by a regular geometric arrangement of equally spaced points. The simplest examples of these are square numbers, 1, 4, 9, 16, which can be represented by squares of side 1, 2, 3 and 4. Other geometric shapes which can lead to special numbers are the triangular shapes and the stellar (star) shapes. Our calculations will be based on the sets of triangular and stellar diagrams that are already provided and those that will be constructed. The aim of this investigation is to examine and determine the general statement for geometric patterns that lead to special numbers as well as to demonstrate a good and clear understanding of patterns and the operations that can be done with them. At the end of the project, we should be able to generate expressions and recognise patterns of other various geometric arrangements.

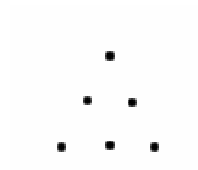
Triangular Pattern:

In the following, I will examine how we can derive general statements of patterns within triangular shapes. To begin, we can use a sequence of diagrams to better illustrate the pattern:

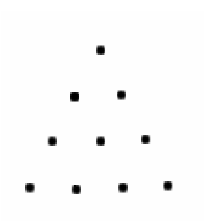
1 = 1



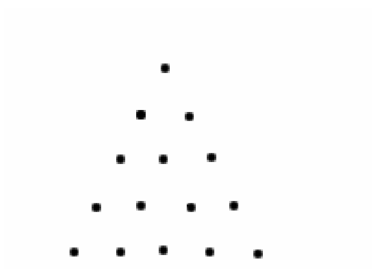
3 = 1+2



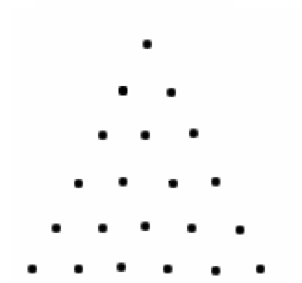
6 = 1+2+3



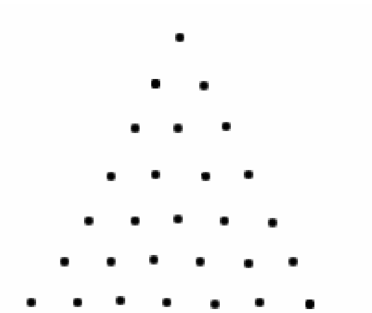
10 = 1+2+3+4



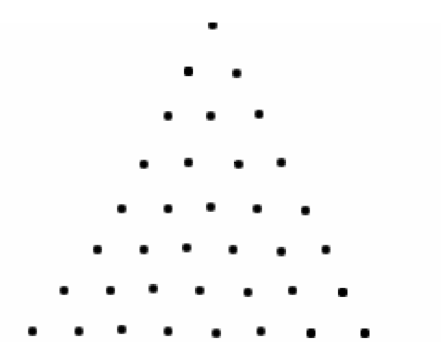
15 = 1+2+3+4+5



21 =



28 = 1+2+3+4+5+6+7



36 = 1+2+3+4+5+6+7+8

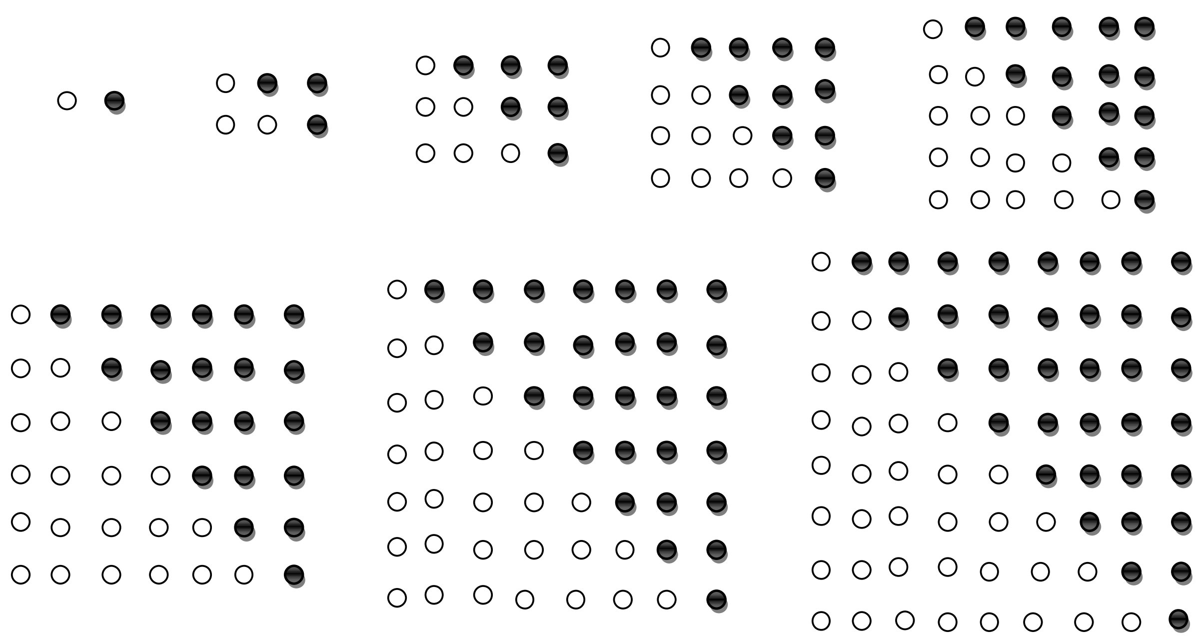
Firstly, looking at the pattern, we can see that the number of dots is the sum of a series of consecutive positive numbers. Considering this pattern within the triangular diagrams, we can use it to formulate the general formula. In the Math 10 pure curriculum, we were taught the sum of the series of consecutive 100 positive numbers. Accordingly, the sum of the first 100 positive numbers = 1 + 2 + 3... + 100 = (1+100) + (2+99) + ...+ (50+51) = (50)(101) = 5050. In another form, the sum can be expressed as (50)(100 + 1). By using this information, we can deduce the number of dots in each diagram of the triangular pattern in which follows a similar pattern.

The variables are as follows: tn = triangular number (numbers of dots)

n = the stage

In order to modify the expression for the previous series of 100 consecutive positive numbers into the general formula for the triangular pattern, we can simply replace the number 100 within the expressing (50)(100+1) by n. The reason for which is because within the context of the series of 100 consecutive numbers, there is 100 terms and therefore the 100th stage. Also, as you can see the 50 within the expression is derived by dividing 100 by 2. As such, it can be modified as being (n/2). After the two modifications, we can rewrite the expression for adding consecutive a hundred positive numbers into the general statement: $tn = (n / 2)(n + 1)$. After simplifying this multiplication of a monomial by a binomial, we get the general statement as: $tn = 0.5n^2 + 0.5n$. This general statement can also express the pattern of triangular geometric shapes as it poses the same pattern as the series of 100 consecutive positive numbers.

Nevertheless, there is also an entirely new method of finding a general statement that represents the n^{th} triangular number in terms of n that is enabled by the triangular geometric shape. Considering the follow arrays of rectangular diagrams shown below:



In each diagram, there are two copies of the triangular diagram, a black one and a white one. The original triangular diagrams from stage 1 to stage 8 now transformed into rectangles. It can be realized then that the number of dots of the original triangular diagram can be derived through the simple triangular area formula:

It is easy to see then that: $A = \frac{1}{2}bh$

The 1^{st} triangular number= (1 X 2) / 2 = 1

The 2^{nd} triangular number= (2 X 3) / 2 = 3

The 3^{rd} triangular number= (3 X 4) / 2 = 6

The 4^{th} triangular number= (4 X 5) / 2 = 10

The 5^{th} triangular number= (5 X 6) / 2 = 15

The 6^{th} triangular number= (6 X 7) / 2 = 21

The 7^{th} triangular number= (7 X 8) / 2 = 28

The 8^{th} triangular number = (8 X 9) / 2 = 36

In general, then:

$$tn = [n \text{ X } (n + 1)] / 2$$

We can simplify the above general statement and arrive at:

$$tn = 0.5n^2 + 0.5n$$

To have a correct general expression, it is essential that the statement be verified. In order to verify, we can use the already known value of 21 dots for the 6^{th} stage. Therefore, simply substitute 6 for *n* into the derived general statement:

$$LS= 0.5(6)^2 + 0.5(6)$$

$$RS= 21$$

$$LS = RS$$

For further insurance, simply substitute 3 for *n* into the general statement:

$$LS= 0.5(3)^2 + 0.5(3)$$

$$RS= 6$$

$$LS = RS$$

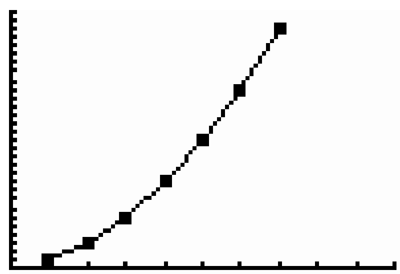
By using advanced technology such as a graphing calculator, we can derive the general statement that represents the n^{th} triangular number in terms of *n* much faster. The following table gives the number of evenly spaced dots in relation to the first seven stages of the pattern.

L3	L4	L5 3
1	1	------
2	3	
3	6	
4	10	
5	15	
6	21	
7	28	

L3(1)=1

By plotting the first seven stages of the pattern on a graph, we can study the relation within the pattern much more effectively.

Graph settings are as follows: X:[0, 10, 1] Y:[0, 30, 1]



The above presents the data from the previous table in graphical form. It illustrates the relation between the stage and the number of evenly spaced dots. The stage of the pattern is positioned on the X-axis and the number of evenly spaced dots on the Y-axis.

From this graph, we can see that the relation of the triangular pattern forms a parabola.

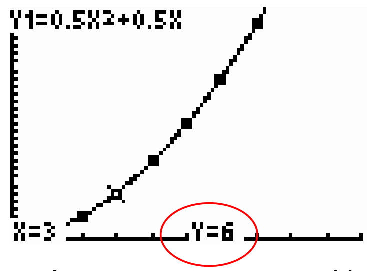
By undertaking regression of the data, we can easily deduce the general statement.

QuadReg
$y=ax^2+bx+c$
$a=.5$
$b=.5$
$c=0$
$R^2=1$

With the resourceful use of calculator, we can easily deduce the general statement as:

$$Y = 0.5x^2 + 0.5x$$

Being a method of verification, we can use the 'value' function of the calculator. Let us take into consideration that the 3rd stage contains 6 numbers of dots as shown in previous diagram. Therefore, if the Y value equals 6 when the X value is 3, then we can conclude that the general statement derived from the calculator is correct.



It is clearly indicated that the general statement generated by the graphing calculator did in fact derive Y as 6 when X is 3. Therefore, the derived general statement through the use of a TI-84 graphing calculator is correct.

Considering that the general statement must represent the n^{th} triangular number in terms of n, the general statement derived from the calculator can be modified into the general statement as : $tn = 0.5n^2 + 0.5n$.

After investigating the pattern of triangular geometric shapes in three different methods that have led to the same results, we can conclude then that the general statement that represents the n^{th} triangular number in terms of n for triangular shapes is:

$$tn = 0.5n^2 + 0.5n$$

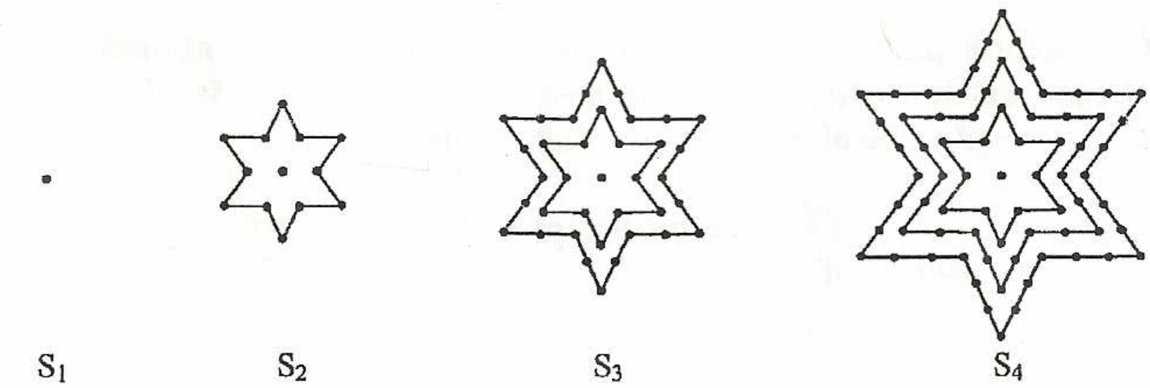
In other words, the n^{th} triangular number is the sum of the half of the n^{th} stage squared and the half of the n^{th} stage.

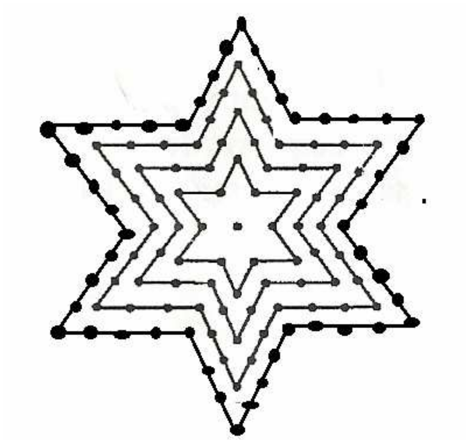
The scopes or limitations of this general statement are :

Domain: $\{n \geq 1 \mid n \in N\}$, Range: $\{tn \geq 1 \mid tn \in N\}$.

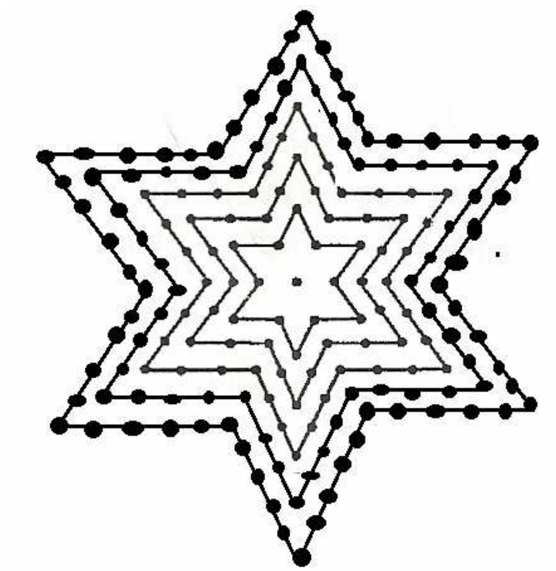
Stellar Shapes Pattern:

After the investigation of the triangular patterns, we will now consider a more complicated geometric shape which leads to special numbers. In the following, we will investigate the general statement for star shapes with p vertices, leading to p-stellar numbers (number of dots). The first six representations for a star with six vertices are shown in the six stages S_1-S_6 below to better illustrate the pattern. The stellar number at each stage is the total number of dots in the diagram.





S_5



S_6

The stellar number as a relation to the first six stages of the sequence is as follows:

Stage	Stellar Number
1	1
2	13
3	37
4	73
5	121
6	181

A pattern can be recognized as:
The difference of stellar number between stage 1 and 2 = 12 =2 X 6
The difference of stellar number between stage 1 and 3 = 36 = 6 X 6
The difference of stellar number between stage 1 and 4 = 72 = 12 X 6
The difference of stellar number between stage 1 and 5 = 120 = 20 X 6
The difference of stellar number between stage 1 and 6 = 180 = 30 X 6

The above difference between the stellar numbers of different stages can be utilized to illustrate a relation between the stage number, the number of vertices and the stellar number.

We can realize that the stellar number shown within the table above is one more than its difference. For instance, the stellar number of stage 3 is 37, which is one more than the difference between stages 1 and 3, which is 36.

The relation is illustrated below: the difference derived earlier is shown in brackets
The stellar number of S_2 = (2 X 6) + 1 = (2 X 1 X 6) + 1 = 13
The stellar number of S_3 = (6 X 6) + 1 = (3 X 2 X 6) +1 = 37
The stellar number of S_4 = (12 X 6) + 1 = (4 X 3 X 6) + 1= 73
The stellar number of S_5 = (20 X 6) + 1 = (5 X 4 X 6) + 1 = 121
The stellar number of S_6 = (30 X 6) + 1 = (6 X 5 X 6) + 1 = 181

These statements above can be utilized to invent the general statement:

S_n = stellar numbers (number of dots)
n = stage number

Taking the previously derived statement ‘The stellar number of S_3= (3 X 2 X 6) +1 = 37’ into account, we can replace the 3 by n, replace 2 by (n-1) and replace 37 by S_n.

Therefore, we arrive to the general formula as:

$$S_n = (n)(n-1)(6) + 1$$
$$= 6n^2 – 6n + 1$$

Since we already know that there are 121 dots for the 5th stage, we can verify the above general statement for six vertices stellar shape by simply substitute 5 for n into the general statement:

$$\text{The } 5^{th} \text{ stage stellar number} = 6(5)^2 – 6(5) + 1= 121$$

Considering that the previous general formula for the triangular pattern is a quadratic function, it is reasonable to assume then that this similar style of pattern will also result in a quadratic statement. Therefore, there is a second method of deriving the general statement for a stellar shape with 6 vertices.

Without the use of a graphing calculator, we can find the quadratic regression of the values obtained from the diagrams to find the general statement.

Thus, we need three sets of values: X-value = n^{th} stage Y-value = stellar number

(1, 1) (2, 13) (3,37)

Using these points, we can generate a statement in the general form: $y = ax^2 + bx + c$

$1 = a(1)^2 + b(1) + c$

$13 = a(2)^2 + b(2) + c$

$37 = a(3)^2 + b(3) + c$

$1 = a + b + c$	$13 = 4a + 2b + c$	$-12 = (-3)a - b$
$\underline{13 = 4a + 2b + c}$	$\underline{37 = 9a + 3b + c}$	$\underline{-24 = (-5)a - b}$
$-12 = (-3)a - b$	$-24 = (-5)a - b$	$12 = 2a$

$a = 6$

$b = (-3)(6) + 12$

$= -6$

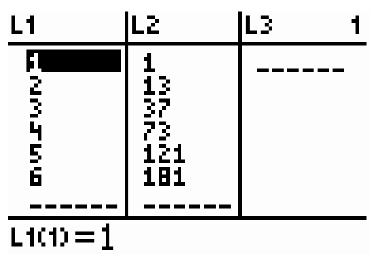
$c = 13 - 4(6) - (2)(-6)$

$= 1$

In conclusion, the general formula derived from using the second method in the general form is: $y = 6x^2 - 6x + 1$. In order to keep the general statement consistent with the general statement derived using the first method, we must express the general statement as the S_n (stellar numbers) in terms of n (stage number). As such, the general statement becomes: $S_n = 6n^2 - 6n + 1$

Now, as a third method, by using a TI-84 Plus Graphing Calculator, we can derive the general expression much more time efficiently and conveniently.

The following table gives the number of dots (i.e. the stellar number) in relation to the first 7 stages of the pattern.

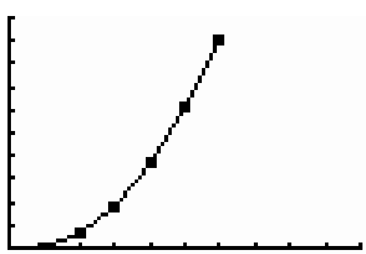


L1	L2	L3 1
1	1	------
2	13	
3	37	
4	73	
5	121	
6	181	
------	------	

L1(1)=1

The result of plotting the data above is shown below:

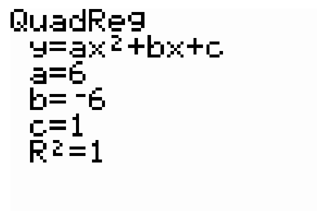
Graph settings are the following: X:[0, 10, 1] Y:[0, 200, 20]



The n^{th} stage is positioned on the X-axis and the stellar number on the Y-axis.

The main observation that can be made from this graph is that it is in the shape of a parabola similar to the previous graph that modeled the triangular pattern.

Through the use of QuadReg function in the calculator, the general statement for the 6-stellar number pattern can be derived:

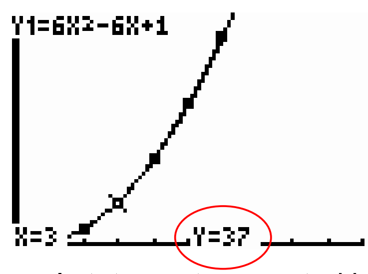


Since the value of R^2 is 1, this indicates that this general expression exactly fits to the stellar pattern.

Therefore, the graph generated by the graphing calculator can be exactly modelled by using the following quadratic function:

$$Y = 6x^2 - 6x + 1$$

We can again use the 'value' function of the calculator as a method of verification. Let us take into consideration that the 3rd stage contains 37 numbers of dots as shown in the previously shown table. Therefore, if the Y value equals 37 when the X value is 3, then we can conclude that the general statement derived from the calculator is correct.



It is clearly indicated that the general statement generated by the graphing calculator did in fact derive Y as 37 when X is 3. Therefore, the derived general statement through the use of a TI-84 graphing calculator is correct.

After altering Y with S_n (stellar number) and X with *n (stage number)*, the general expression for a 6 vertices stellar shape becomes:

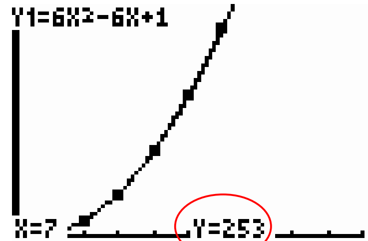
$$S_n = 6n^2 – 6n + 1$$

From this general statement, we can find an expression for the 6-stellar number beyond stage 6, such as stage 7 by replacing n with 7.

$$S_7 = 6(7)^2 – 6(7) + 1$$

$$= 253$$

We can verify the validity of this expression by using the graphing calculator.



With the use of the Value function of the calculator, we derived the Y value as 253 when X is 7. In other words, the expression for the 6-stellar number at stage S_7 is valid.

The same method used to find 6-stellar number at stage S_7 can also be used to find the stellar number for stage S_{23} and even up to stage S_{100}.

$$S_{23} = 6(23)^2 - 6(23) + 1 = 3037$$

$$S_{100} = 6(100)^2 - 6(100) + 1 = 59401$$

The use of the general statement to determine the stellar number provided much needed convenience when dealing with stage S_{23} and especially stage S_{100}.

Recognizing that the coefficient of 6 in the general statement '$S_n = 6n^2 - 6n + 1$' represents the number of vertices, we can derive the general statement for other values of vertices (p) by altering the coefficients.

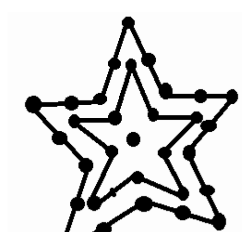
Therefore, for stellar shapes with 5 vertices, the general statement then becomes:

$$Sn = 5n^2 - 5n + 1$$

In order to verify, we can simply use diagrams of stellar shape with 5 vertices:

S1

S2



S3

In order to affirm the validity of the general statement for stellar shapes with 5 vertices, the number of dots counted must equal the number of dots derived from the statement:

LS = number of dots derived from the statement

RS = number of dots counted in the diagram

Stage 1	Stage 2	Stage 3
LS= $5(1)^2 - 5(1) + 1$	LS= $5(2)^2 - 5(2) + 1$	LS= $5(3)^2 - 5(3) + 1$
RS= 1	RS= 11	RS= 31
LS = RS	LS = RS	LS = RS

Hence, the overall general statement, in terms of *p* and *n*, which generates the sequence of *p*-stellar numbers for any value of *p* at stage *Sn* is:

$$S_n = pn^2 - pn + 1$$

In order to test the validity of this general statement, we can insert numbers in place of the variables and the parameters. In knowing that the stellar number of stage 3 for a star shape with 6 vertices is 37; therefore, the stellar number derived from this formula under the same circumstances should be the same.

LS = 37

$p = 6$, $n = 3$

RS = $pn^2 - pn + 1$

$= 6(3)^2 - 6(3) + 1 = 37$

LS = RS

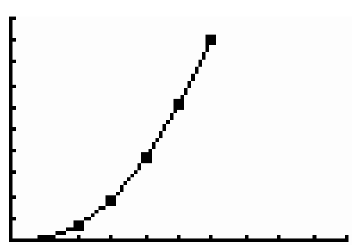
Therefore, the general statement is valid.

To determine the scope or limitations, we can begin by looking at the graph:

The graph settings are:

X:[0, 10, 1]

Y:[0, 200, 20]



The scope or limitations of the general statement in terms of geometric patterns are: Domain: $n \geq 1 \square n \in N$, Range: $Sn \geq 1 \square Sn \in N$. When examining the general statement in terms of a geometric shape, the stage cannot be smaller than 1. Because this means that when stage is 0, the number of dots will become 1. "$6(0)^2 – 6(0) + 1= 1$". This is invalid and wrong because when there is no stage there shouldn't be any figure, therefore zero dots. As well, it is also impossible to have a negative stage of a geometric pattern. However, the stage can be any positive natural number for it can be 3, or even 1000. In addition, notice that the graph that models the pattern begins when x= 1, rather than from the origin. With these reasons, when n must be smaller and equal than 1; and is a set of natural numbers in order to make the general statement valid. When considering the range, since the stage must start at 1, therefore, the number of dots will also begin at 1. In addition, having a negative number of dots is also invalid. Therefore, Sn is limited to be greater and equal than 1, and is also a set of natural numbers. Since p is a parameter, it generally does not have a limitation. However, in the case of a geometric pattern, $p > 0$ since we cannot have a negative number of vertices in a geometric shape. As well, p cannot be 0 because the general formula would not have worked as the stellar number of every stage will become 1. To illustrate, when we substitute 0 for p, the general statement becomes $0n^2 – 0n + 1 = 1$. No matter what n is, the result will always be 1. Therefore, the derived general formula is invalid in a situation where p is 0.

Conclusion:

Through analysing the patterns, the resourcefully use of a graphing calculator, the studying of the graphs and undertaking regression of the data, we can easily deduce the general statement of the two geometric patterns. The general statement of the triangular pattern is $tn = 0.5n^2 + 0.5n$ where tn is the number of evenly spaced dots, and n is the n^{th} stage. The general statement of the stellar pattern is $S_n = pn^2 – pn + 1$ where S_n is the stellar number (number of dots), p is the number of vertices, and n is the n^{th} stage. From the above investigation, we come to a conclusion that the general statements for any geometric shapes which lead to special numbers are quadratic functions with an upward concaving parabola that is only valid in the first quadrant of the graph where the Domain: $n \geq 1 \square n \in N$, and Range: $Sn \geq 1 \square Sn \in N$.