Introduction

In this investigation, the use of the properties of logarithms will be used to identify patterns, relationships, and limits of logarithms and sequences. A logarithm is an exponent in which the base must have in order to receive the number. All values found will be in the form p/q to further show a relationship with the terms.

Part 1

In the first sequence given, when examined closely, you will see that the log number remains the same throughout the part of the sequence given. You will also notice that it is the base of the logarithms that is altered.

$\log_2 8, \log_4 8, \log_8 8, \log_{16} 8, \log_{32} 8, \ldots$

To begin, it would prove to be useful to figure out what sort of sequence it really is. By looking that the logarithms, it is safe to conclude that the sequence given cannot be an arithmetic sequence; the reason being that the bases of each logarithm are different, therefore, there is no common difference that describes the sequence as arithmetic.

Since it has been proven that the sequence cannot be arithmetic, the only type of sequence left, is a geometric sequence. A geometric sequence is a sequence whose consecutive terms are multiplied by a fixed, non-zero real number called a common ratio. Therefore, to prove that the sequence is indeed, geometric, the common ratio must be found.

When the sequence is examined closely, you will notice that the log number remains the same throughout the sequence and that it is the base of the logarithm that changes with every term. Understanding this, you can use the bases to find the common ratio of the sequence by dividing the consecutive terms by the terms preceding them.

After doing this, it is found that the common ratio of the sequence is 2. The process of evaluating the expression thus can begin…

$\log_2 8, \log_4 8, \log_8 8, \log_{16} 8, \log_{32} 8, \ldots$

$4/2 = 2$, $8/4 = 2$, $16/8 = 2$, $32/16 = 2$

One way of evaluating what the values of the next two terms are is by using the formula $\log_{m^n} m^k$ where m is the base and number (constant number), k is the exponent in which m is raised, and n is the term number

$\log_2 8 = \log_{2^1}(2^3)$, $\log_4 8 = \log_{2^2}(2^3)$, $\log_{16} 8 = \log_{2^4}(2^3)$, $\log_{32} 8 = \log_{2^5}(2^3)$

After finding this, you can use the Change-of-Base formula to convert the logarithm into a ratio between the two components of the logarithm.

$\log_2{}^1(2^3) = \log 2^3 / \log 2^1$
$\log_2{}^2(2^3) = \log 2^3 / \log 2^2$
$\log_2{}^4(2^3) = \log 2^3 / \log 2^4$
$\log_2{}^5(2^3) = \log 2^3 / \log 2^5$

Then, according to the power property of logarithms, the exponent of a logarithm can be moved to the front of the log as a multiplier which can be further simplified by canceling out log2 on each side of the fraction. This gives the values in the form p/q.

$\log 2^3 / \log 2^1 = 3\log 2 / 1\log 2 = 3/1$
$\log 2^3 / \log 2^2 = 3\log 2 / 2\log 2 = 3/2$
$\log 2^3 / \log 2^3 = 3\log 2 / 3\log 2 = 3/3 = 1$
$\log 2^3 / \log 2^4 = 3\log 2 / 4\log 2 = 3/4$
$\log 2^3 / \log 2^5 = 3\log 2 / 5\log 2 = 3/5$

After evaluating the logarithms, you can see that the denominator of each value is the same as the term number. Therefore the conclusion that the value of the sequence can be calculated by using the expression $3/n$ where n is the term number. Using this expression, the value of the 6th and 7th terms may be found.

Value of 6th term	Value of 7th term	
3/6	3/7	

Multiply the ratios by log2 and rewrite the logarithm using Change-the-Base.

$3\log 2 / 6\log 2$	$3\log 2 / 7\log 2$	multiply log2 to both sides
$\log 2^3/ \log 2^6$	$\log 2^3 / \log 2^7$	change multiplier to exponent
$\log 8 / \log 64$	$\log 8 / \log 128$	expand exponential
$\log_{64} 8$	$\log_{128} 8$	change the base to rewrite logarithm

In the second, third, and fourth sequences, the setup of the terms is very similar to that of the first sequence where the base of the logarithm changes but not the . Because of that, the sequences are also both geometric sequences. Finding the common ratio can further prove this theory to be correct.

Second Sequence…
$\log_3 81$, $\log_9 81$, $\log_{27} 81$, $\log_{81} 81$, …

9/3 = 3 , 27/9 = 3 , 81/27 = 3 the common ratio of the sequence is 3

Third Sequence…

$\log_5 25$, $\log_{25} 25$, $\log_{125} 25$, $\log_{625} 25$, …

25/5 = 5 , 125/25 = 5 , 625/125 = 5   the common ratio of the sequence is 5

Fourth Sequence…

$\log_m m^k$, $\log_{m^2} m^k$, $\log_{m^3} m^k$, $\log_{m^4} m^k$, …

$m^2/m = m$, $m^3/m^2 = m$, $m^4/m^3 = m$ the common ratio of the sequence is m

Now you can evaluate the terms in the form p/q.

Second Sequence

nth term	Corresponding logarithm	Rewrite into exponential form	Change the base	Change the exponent into a multiplier of logarithm	Simplify ratio
1	$\log_3 81$	$\log_{3^1}(3^4)$	$\log 3^4 / \log 3^1$	4log3 / 1log3	4/1
2	$\log_9 81$	$\log_{3^2}(3^4)$	$\log 3^4 / \log 3^2$	4log3 / 2log3	4/2 = 2
3	$\log_{27} 81$	$\log_{3^3}(3^4)$	$\log 3^4 / \log 3^3$	4log3 / 3log3	4/3
4	$\log_{81} 81$	$\log_{3^4}(3^4)$	$\log 3^4 / \log 3^4$	4log3 / 4log3	4/4 = 1
5	$\log_{243} 81$	$\log_{3^5}(3^5)$	$\log 3^4 / \log 3^5$	4log3 / 5log3	4/5
6	$\log_{729} 81$	$\log_{3^6}(3^6)$	$\log 3^4 / \log 3^6$	4log3 / 6log3	4/6 = 2/3

As the sequence shows, the general expression used to obtain the answer is 4/n.

Value of 5th term	Value of 6th term	
4/5	4/6	
4log3 / 5log3	4log3 / 6log3	multiply log3 to both sides
$\log 3^4 / \log 3^5$	$\log 3^4 / \log 3^6$	change multiplier into exponent
log81 / log243	log81 / log729	expand exponential
$\log_{243} 81$	$\log_{729} 81$	change the base to rewrite logarithm

Third Sequence

nth term	Corresponding logarithm	Rewrite into exponential form	Change the base	Change the exponent into a multiplier of logarithm	Simplify ratio

1	$\log_5 25$	$\log_{5^1}(5^2)$	$\log 5^2 / \log 5^1$	2log5 / 1log5	2/1 = 2
2	$\log_{25} 25$	$\log_{5^2}(5^2)$	$\log 5^2 / \log 5^2$	2log5 / 2log5	2/2 = 1
3	$\log_{125} 25$	$\log_{5^3}(5^2)$	$\log 5^2 / \log 5^3$	2log5 / 3log5	2/3
4	$\log_{625} 25$	$\log_{5^4}(5^2)$	$\log 5^2 / \log 5^4$	2log5 / 4log5	2/4 = 1/2
5	$\log_{3125} 25$	$\log_{5^5}(5^2)$	$\log 5^2 / \log 5^5$	2log5 / 5log5	2/5
6	$\log_{15625} 25$	$\log_{5^6}(5^2)$	$\log 5^2 / \log 5^6$	2log5 / 6log5	2/6 = 1/3

As the sequence shows, the general expression used to obtain the answer is 2/n.

<u>Value of 5th term</u>	<u>Value of 6th term</u>	
2/5	2/6	
2log5 / 5log5	2log5 / 6log5	multiply log3 to both sides
$\log 5^2 / \log 5^5$	$\log 5^2 / \log 5^6$	change multiplier into exponent
log25 / log3125	log25 / log15625	expand exponential
$\log_{3125} 25$	$\log_{15625} 25$	change the base to rewrite logarithm

<u>Fourth Sequence</u>

nth term	Corresponding logarithm	Rewrite into exponential form	Change the base	Change the exponent into a multiplier of logarithm	Simplify ratio
1	$\log_m m^k$	$\log_{m^1} m^k$	$\log m^k / \log m^1$	klogm / 1logm	k/1
2	$\log_{m^2} m^k$	$\log_{m^2} m^k$	$\log m^k / \log m^2$	klogm / 2logm	k/2
3	$\log_{m^3} m^k$	$\log_{m^3} m^k$	$\log m^k / \log m^3$	klogm / 3logm	k/3
4	$\log_{m^4} m^k$	$\log_{m^4} m^k$	$\log m^k / \log m^4$	klogm / 4logm	k/4
5	$\log_{m^5} m^k$	$\log_{m^5} m^k$	$\log m^k / \log m^5$	klogm / 5logm	k/5
6	$\log_{m^6} m^k$	$\log_{m^6} m^k$	$\log m^k / \log m^6$	klogm / 6logm	k/6

To evaluate the 5th and 6th terms in the sequence, only the n in the formula $\log_{m^n} m^k$ needs to be replaced because the sequence's logarithms are already in the exact same form.

So due to the information provided by the four sequences, we can come to the conclusion that the formula of $\log_{m^n} m^k$ can be replaced with the expression k/n where k is a constant.

Below is a graph showing the values of the three sequences

Part 2

Given the following sequences, there is a relationship between the 3 terms in each sequence.

$\log_4 64$, $\log_8 64$, $\log_{32} 64$
$\log_7 49$, $\log_{49} 49$, $\log_{343} 49$
$\log_{1/5} 125$, $\log_{1/125} 125$, $\log_{1/625} 125$
$\log_8 512$, $\log_2 512$, $\log_{16} 512$

When closely examined, you will notice that the first logarithm's base multiplied by the second logarithm's base results in the third logarithm's base.

First Sequence
(4)(8) = 32

Second Sequence
(7)(49) = 343

Third Sequence
(1/5)(1/125) = 1/625

Fourth Sequence
(8)(2) = 16

Values found using the base numbers of the first and second logarithms.

Different values can be used for the n^{th} terms in a sequence. Below will be two examples of sequences that follow this same pattern.

Sequence #1
$\log_6 216$, $\log_3 216$, $\log_{18} 216$

(6)(3) = 18

Sequence #2
$\log_2 32$, $\log_5 32$, $\log_{10} 32$

(2)(5) = 10

Multiply the bases of the first and second logarithms to find the base of the third logarithms.

Part 3

Given that $\log_a x = c$ and $\log_b x = d$, find a general statement that expresses $\log_{ab} x$ in terms of c and d.

$\log_a x = c$, $\log_b x = d$	
$(a^c)^d = x^d$, $(b^d)^c = x^c$	rewrite the logarithms as exponential form
$a^{cd} = x^d$, $b^{cd} = x^c$	multiply the exponents
$(ab)^{cd} = x^{(c+d)}$	multiply the two equations together
$(cd)\log ab = (c+d)\log x$	rewrite exponentials as logarithms
$(cd)\log ab/c+d = \log x$	divide c+d by both sides to get logx by itself
$cd/c+d = \log x / \log ab$	divide logab by both sides
$cd/c+d = \log_{ab} x$	rewrite the right half of the equation as a logarithm thus producing the general statement that expresses $\log_{ab} x$ in terms of c and d

As the work shows, cd/c+d is the general statement needed to be found. We can test the validity of this statement by using various values for a, b, and x. For example…

$a = 2$, $b = 4$, $x = 8$

$\log_a x = \log_2 8$ Write the three logarithms needed to make the sequence
$\log_b x = \log_4 8$
$\log_{ab} x = \log_8 8$

$\log_{2^1}(2^3)$, $\log_{2^2}(2^3)$, $\log_{2^3}(2^3)$ Rewrite the logarithms in exponential form

3 , 3/2 , 1 Change the bases, change the exponent due to the power property of logarithms, and simplify

$c = 3$, $d = 3/2$

cd/c+d

= (3)(3/2)/(3+3/2)

=1 the expression cd/c+d is valid because the result is the same as the result of $\log_{ab}x$, which was shown 2 steps before. (1=1)

Even now when it has been proven that the expression cd/c+d is equivalent to the expression $\log_{ab}x$, there are still some things that need to be set in order. There are a series of limitations on these expressions that must be shown so as not to have any values that make the expression false.

Limitations

The first limitation on cd/c+d = $\log_{ab}x$ is that ab>0, the reason being because a logarithm cannot be negative. For example if…

a = -1 b = 5 x = 125

$\log_{ab}x$

= $\log_{(-1)(5)}125$ substitute the values in for a, b, and x

= $\log_{-5}125$ combine

= log125/log-5 by using change-of-base, you can clearly see the negative logarithm which does not exist

Another limitation of cd/c+d = $\log_{ab}x$ is that ab $\neq$ 1. This cannot be because log1 is equal to zero. If…

a = 1 b = 1 x = 125

$\log_{ab}x$

= $\log_{1}125$ substitute in the values for a, b, and x

= log125 / log1 change-of-base to be able to see the work of the equation easily

=log125/ 0 solve log1, which equals to 0.

Yet another limitation of cd/c+d = $\log_{ab}x$ is that ab $\neq$ 0. This cannot be because log0 is not possible. If…

a = 0 b = 5 c = 125

$\log_{ab}x$

= $\log_{(0)(5)}125$ substitute in the values for a, b, and x

$= \log_0 125$	combine
$= \log 125 / \log 0$	change-of-base to be able to see the work of the equation easily.

Not possible since log0 does not exist.

The next limitation of $cd/c+d = \log_{ab}x$ is that $x>0$. It is a limitation for the same reasons as the first limitation: a logarithm cannot be negative.

The last limitation on $cd/c+d = \log_{ab}x$ is that $x \neq 0$. This limitation is also a limitation for the same reasons as the second limitation: a logarithm cannot equal zero.

Therefore, from the information provided, you can conclude that the general statement works only with positive numbers.

Conclusion

In conclusion, from the information in this investigation, we can say that the value of the logarithmic sequences in the form of p/q can be reached through the equation k/n. Regularly, the value of logarithmic sequences can be reached through the equation $\log_{ab}x$. We also have determined what the general statement was to the expression $\log_{ab}x$. The general statement is cd/c+d which has a few limitations that are $ab>0$, $ab \neq 1$, $ab \neq 0$, $x>0$, and $x \neq 0$.