Logarithm Bases

In the beginning of this problem, we are given multiple sequences of logarithms, and are told to write down the next two terms of each sequence. Here is a table of the first sequence, including the next two terms and the numerical equivalence of each term:

# of Term	1	2	3	4	5	6	7
Term	$\log_2 8$	$\log_4 8$	$\log_8 8$	$\log_{16} 8$	$\log_{32} 8$	$\log_{64} 8$	$\log_{128} 8$
Numerical Equivalence	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$

I created the last two terms in this sequence, terms 6 and 7, simply by doubling the base of the logarithm for each term. In more proper mathematical terms, I used the formula 2^n, where n is the number of the term. I noticed that each numerical equivalence seems to have 3 as the numerator, and the number of the term as the denominator. In this manner, I created a formula to find the numerical equivalence for the nth term of the sequence in the form $\frac{p}{q}$, where both p and q are integers: $y = \frac{3}{n}$.

I used this method of investigation for the other two sequences as well. Here are tables for each of the two sequences:

# of Term	1	2	3	4	5	6
Term	$\log_3 81$	$\log_9 81$	$\log_{27} 81$	$\log_{81} 81$	$\log_{243} 81$	$\log_{729} 81$
Numerical Equivalence	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$

# of Term	1	2	3	4	5	6
Term	$\log_5 25$	$\log_{25} 25$	$\log_{125} 25$	$\log_{625} 25$	$\log_{3125} 25$	$\log_{15625} 25$
Numerical Equivalence	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$

I noticed that in both of these sequences as well, the numerical equivalences seem to have a constant integer as the numerator, and the number of the term as the denominator. Therefore, the formula for the first sequence is $y = \frac{4}{n}$, and the formula for the second sequence is $y = \frac{2}{n}$.

Now, we are asked to calculate a set of many logarithms, in the form $\frac{p}{q}$, where p and q are both integers. The set is as follows:

Logarithm	$\log_4 64$	$\log_8 64$	$\log_{32} 64$
Numerical Equivalence	$\frac{3}{1}$	$\frac{2}{1}$	$\frac{6}{5}$
Logarithm	$\log_7 49$	$\log_{49} 49$	$\log_{343} 49$
Numerical Equivalence	$\frac{2}{1}$	$\frac{1}{1}$	$\frac{2}{3}$
Logarithm	$\log_{\frac{1}{5}} 125$	$\log_{\frac{1}{125}} 125$	$\log_{\frac{1}{625}} 125$
Numerical Equivalence	$\frac{-3}{1}$	$\frac{-1}{1}$	$\frac{-3}{4}$
Logarithm	$\log_8 512$	$\log_2 512$	$\log_{16} 512$
Numerical Equivalence	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{9}{4}$

Finding the numerical equivalences of the logarithms was very easy; I simply calculated them using a graphing calculator. The challenging part arose when I was asked to find a way to obtain the third answer in each row from the first two answers in each row. I was also told to let $\log_a x = c$ and $\log_b x = d$, and to find a general statement that expresses $\log_{ab} x$, in terms of c and d. These two questions are mostly the same. The second one is basically asking me to find a property that allows me to combine two logarithms with the same x-value. The first question is asking me to calculate the third logarithm in each row using the answers to the first two logarithms. Since the base of the third logarithm in each row is the product of the first two in each row, both of the questions are asking the same thing.

After making a table of the numerical equivalences of all of the logarithms, I began studying it and looking for a pattern. After much thinking, I realized that the numerator of the third logarithm is equal to the product of the first two logarithms. I also noticed that the denominator of the third logarithm is equal to the sum of the first two logarithms. Using this pattern and logic, I came to a conclusion:

$$\log_{ab} x = \frac{cd}{c+d}.$$

This is my general statement. I can use this property to solve the third logarithm in each row using the answers to the first two logarithms in each row.

I was also asked to create two more examples that fit the pattern that I discovered. Here are two more sets of three logarithms:

$\log_2 32$	$\log_8 32$	$\log_{16} 32$
$\log_{10} 2$	$\log_{12} 2$	$\log_{120} 2$

Both of these sets fit the property and pattern.

I was then asked to define the scope and limitations of my general statement. In order to discover this, I tested the formula using different values for a, b, and x. First, I made a=0, and picked random values for b and x.

Logarithm	$\log_0 2$	$\log_2 2$	$\log_0 2$
Numerical Equivalence	ERROR	1	ERROR

Obviously, it didn't work. First of all, since the base of the third logarithm has to be the product of the bases of the first two logarithms, the third base ended up being 0. Secondly, it is impossible to find the logarithm of 0, or any negative number. Therefore, $a > 0$. I can extend this finding to b, because the base of the third logarithm would be 0 if either of the first two bases were 0. So, $b > 0$.

After testing the formula with zeros, I decided to think about ways to make the formula itself undefined. By undefined, I mean concepts like taking the square root of a negative, dividing by zero, etc. Since there are no square roots involved in this formula, I proceeded to find a way to make the denominator of the formula equal to 0. The only way to do this is to make c and d opposites (negatives) of each other. For example, if c=5 and d=-5, c+d=0 and the formula becomes undefined. In order to make c and d opposites of each other, the bases of the first two logarithms would need to be reciprocals of each other. I tested this theory by using new values for a, b, and x:

Logarithm	$\log_5 25$	$\log_{\frac{1}{5}} 25$	$\log_1 25$
Numerical Equivalence	2	-2	ERROR

This didn't work, because c+d=0, and the general formula ended up with a 0 on the bottom. When I tried to solve the third logarithm using my calculator, it gave me an error message saying "Divide by 0". Therefore, $a \neq \frac{1}{b}$ and $b \neq \frac{1}{a}$.

One more thing that I thought of was that x must be greater than 0, because of the definition of a logarithm. There is no possible way to get any number to equal 0 or any negative number using only exponents. Therefore, $x > 0$.

I couldn't think of any more limitations of the general statement. Here are the limitations that I came up with:

$a > 0$ $b > 0$ $x > 0$ $a \neq \frac{1}{b}$ $b \neq \frac{1}{a}$

I will now repeat my general statement. If we let $\log_a x = c$ and $\log_b x = d$, then

$$\log_{ab} x = \frac{cd}{c+d}.$$

This statement only works with the limitations stated previously. It took me a lot of staring at the numbers and trying different values for a, b, and x, but eventually I discerned a pattern and arrived at my general statement.