Math portfolio

Find the nth term of a logarithmic sequence

In this portfolio I will investigate how we can use changes in the base of a logarithmic sequence to find a general expression. We start with the fallowing logarithmic sequences;

- $\log_2 8, \ \log_4 8, \ \log_8 8, \ \log_{16} 8, \ \log_{32} 8, \log_{64} 8, \ \log_{128} 8$
- $\log_3 81, \ \log_9 81, \ \log_{27} 81, \ \log_{81} 81, \log_{243} 81, \log_{729} 81$
- $\log_5 25, \ \log_{25} 25, \ \log_{125} 25, \ \log_{625} 25, \log_{3125} 25, \log_{15625} 25$

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- $\log_m m^k, \ \log_{m^2} m^k, \ \log_{m^3} m^k, \log_{m^4} m^k, \log_{m^5} m^k, \log_{m^5} m^k$

We can use the last sequence of the sequences above to show a pattern of the change of base in the sequences. By using this information we can deduce the nth term of any logarithmic sequence. We can also see that the last sequence is written with the base and the number if the logarithm both using m raised to a power, but the number of the logarithm, m^k, is the constant in the expression. If we look at the pattern of the other sequences we can see that their number is also constant. The m exponent increases proportionally to the term for each term it is an increase by 1.

Now that we have m both as a base and as the number of the logarithm we can use the change of base rule to simplify the terms:

$$\log_b A = \frac{\log_c A}{\log_c A} \qquad \log_m m^k = \frac{\log_c m^k}{\log_c m}$$

Using the rules for logarithms we get the expression: $\log_m m^k = \frac{k \log_c m}{1 \log_c m}$

The use of m both in the base and the number of the logarithm is a smart thing to do. Because the $\log_c m \ and \ \log_c m^k$ in the change of base rule we can simply cancel ou t the expressions with each other and that will leave us with $\frac{k}{1}$. Then the sequence will look like $\frac{k}{1}, \frac{k}{2}, \frac{k}{3}, \frac{k}{4}, \frac{k}{5}$. This illustrate that k in the nominator is constant and the denominator will increase by 1. Now both n and k is an element of any re al number. This gives us an general statement for the nth term of the sequence.

If we convert the first sequence to the form of $\log_m m^k$, we then get;

$\log_2 2^3$, $\log_{2^2} 2^3$, $\log_{2^3} 2^3$, $\log_{2^4} 2^3$, $\log_{2^5} 2^3$ We can see that the 2's in the expression match the m in the expression. It is easier to change the expression to the $\frac{k}{n}$ form. We can use the change of base rule to convert the expression as you can see below.

$$\log_{2^n} 2^3 = \frac{\log_{10} 2^3}{\log_{10} 2^n} = \frac{3 \log_{10} 2}{n \log_{10} 2} = \frac{3}{n}$$

We end up with a simple general statement for the n th term of the sequence. If we want to find, let's say the 11th term, we simply replace n with 11. Now that we have seen how to find the general statement for the n th term of a sequence, we can do the same to the two remaining sequence using the same techniques.

$$\log_{20^n} 2^3 = \frac{\log_{20} 2^3}{\log_{20} 2^n} = \frac{3 \log_{20} 2}{n \log_{20} 2} = \frac{3}{n}$$

Second sequence:

$\log_3 81$, $\log_9 81$, $\log_{27} 81$, $\log_{81} 81$, $\log_{243} 81$, $\log_{729} 81$

We only have to look at the first term to find the general statement; we can use the remaining terms to check that our $\log_m m^k$ form is correct. You can find the general term using any two consecutive terms from the sequence.

$\log_3 81 = \log_{3^n} 3^4$ Using the change of base rule gives us $\frac{4}{n}$

Third sequence:

$\log_5 25$, $\log_{25} 25$, $\log_{125} 25$, $\log_{625} 25$, $\log_{3125} 25$, $\log_{15625} 25$

$\log_5 25 = \log_{5^n} 5^2$ Using the change of base rule gives us $\frac{2}{n}$

As you may have noticed in these examples, once you have written the term in the form $\log_{m^n} m^k$, you can take the exponent of the number of the logarithm and the exponent of the base and write them as a fraction with the exponent of the base as the denominator.

Now using Microsoft Office Excel I will justify these answers by calculating the term as it is shown and the $\frac{k}{n}$ form that we found. I have chosen to use sequence 1 and 3, from these it is easy to see that what we have done so far is justified:

Justification sequence 1:

Number of logarithm	base	log		k	n	k/n
8	2	3		3	1	3
8	4	1,5		3	2	1,5
8	8	1		3	3	1
8	16	0,75		3	4	0,75
8	32	0,6		3	5	0,6
8	64	0,5		3	6	0,5
8	128	0,428571429		3	7	0,42857143
8	256	0,375		3	8	0,375
8	512	0,333333333		3	9	0,33333333
8	1024	0,3		3	10	0,3

Justification sequence 2:

number of logarithm	base	log		k	n	k/n
25	5	2		2	1	2
25	25	1		2	2	1
25	125	0,666666667		2	3	0,66666667
25	625	0,5		2	4	0,5
25	3125	0,4		2	5	0,4
25	15625	0,333333333		2	6	0,33333333
25	78125	0,285714286		2	7	0,28571429
25	390625	0,25		2	8	0,25
25	1953125	0,222222222		2	9	0,22222222
25	9765625	0,2		2	10	0,2

As you can see the solutions to the term in both forms are exactly the same, in other words, what we have done, writing the terms in the form $\frac{k}{n}$, is a correct way of writing the term, because, as said the solutions are identical. As you can see I have continued the sequence to the tenth term, showing that it is correct not only for the calculations shown earlier. The formulas used in these excel sheets can be viewed in attachments 1 and 2 at the back of the portfolio. Let's move on to the next section.

Applying $\frac{k}{n}$ to another pattern and its use

Now let's try using some of our newly acquired knowledge to solve a problem involving logarithmic sequences similar to those we have just dealt wi th. Take a look at these four sequences:

- $\log_4 64, \log_8 64, \log_{32} 64$
- $\log_7 49, \log_{49} 49, \log_{343} 49$
- $\log_{\frac{1}{5}} 125, \log_{\frac{1}{125}} 125, \log_{\frac{1}{625}} 125$
- $\log_8 512, \log_2 64, \log_{32} 64$

If you look closely at these sequences you will notice that the terms are consecutive, in the three first sequences certain terms have been skipped, and in the last sequence the two first terms have been switched around. Even though these sequences do not seem to have much in common, there is more than meets the eye. It is possible to find the third answer the two first. To get an overview let's rewrite the terms to the form $\frac{k}{n}$.

First sequence:

$\log_4 64 = \log_{2^n} 2^6$ Using the exponent trick gives us $\frac{6}{n}$

There is one difference here from what we did earlier; the base has been changed. This has to be taken into account when finding the placement of the term, as what you now have found is the first term. If we fill in n for each term we end up with, notice that we use $n = 2$ for the first term in the sequence as we have discovered that the actual first term is the one before:

$$n = 2: \frac{6}{2} = 3, \quad n = 3: \frac{6}{3} = 2 \quad n = 5: \frac{6}{5}$$

As you may see now, you can make both $\frac{6}{5}$ using the answers to the two first terms. 6 by multiplying them and 5 by adding them; this gives a formula:

$$\frac{3 \times 2}{3 + 2} = \frac{6}{5}$$

If we try to generalize this formula we start by renaming 2 and 3 as variables. As 2 and 3 are the solution to terms we can use the variable for an arbitrary term in a sequence, U_n. In this case both 2 and three come from terms in the sequence, so we give them an identifier so we do not confuse them. For simplicity we will use a and b as these identifiers. This gives:

$$\frac{U_{n_a} \times U_{n_b}}{U_{n_a} + U_{n_b}}$$

To test this new general expression let us try it on the remaining sequences.

Second sequence:

We have to begin by writing them in the $\frac{k}{n}$ form:

$\log_7 49 = \log_{7^n} 7^2$ Looking at the exponents we have $\frac{2}{n}$. Filling in for n gives us:
$n = 1{:}\ \frac{2}{1} = 2, \quad n = 2{:}\ \frac{2}{2} = 1, \quad n = 3{:}\ \frac{2}{3}$

Using the new formula:
$$\frac{U_{n_a} \times U_{n_b}}{U_{n_a} + U_{n_b}}$$

$$\frac{1 \times 2}{1 + 2} = \frac{2}{3}$$

So far so good, let's do the rest of the sequences:

Third sequence:

$\log_{\frac{1}{5}} 125 = \log_{\frac{1}{5}^n} \frac{1}{5}^{-3}$ Looking at the exponents we have $\frac{-3}{n}$. Filling in for n gives us:
$n = 1{:}\ \frac{-3}{1} = -3, \quad n = 3{:}\ \frac{-3}{3} = -1, \quad n = 4{:}\ -\frac{3}{4}$

Using the formula
$$\frac{U_{n_a} \times U_{n_b}}{U_{n_a} + U_{n_b}}$$

$$\frac{-3 \times -1}{-3 + -1} = -\frac{3}{4}$$

Even though the minus is in front of the 4 and not the 3 it is still the same number, as both will return negative.

Fourth sequence:

$\log_8 512 = \log_{2^n} 2^9$ Looking at the exponents we have $\frac{9}{n}$. Filling in for n gives us:
$n = 3{:}\ \frac{9}{3} = 3, \quad n = 1{:}\ \frac{9}{1} = 9, \quad n = 4{:}\ \frac{9}{4}$

Again using the formula:

$$\frac{U_{n_a} \times U_{n_b}}{U_{n_a} + U_{n_b}}$$

$$\frac{3 \times 9}{3 + 9} = \frac{27}{12} = \frac{9}{4}$$

Now that we have established that the formula works , we can make two more examples of our own just to see that it works. Before doing this it is important that you have noticed the pattern in the formula so far. If you look at the n's that have been used so far you should see the pattern clearly. The sum of the s of the first two terms has always equaled the of the third term. We can quickly discover what happens when this presumption is not fulfilled:

Two more examples to underline this statement:

$Log_2\ 8$ $log_4\ 8$ $log_{16}\ 16$

$Log_8\ 4096$ $log_{64}\ 4096$ $log_{512}\ 4096$

$Log_2\ 16 = \log 16 \div \log 2 = \log 2^4 \div \log 2^2 = 4 \div 2 = 2$

$Log_4\ 16 = \log 16 \div \log 4 = \log 2^4 \div \log 2^3 = 3 \div 2 = 1.33$

$Log_{16}\ 16 = \log 16 \div \log 16 = \log 2^4 \div \log 2^4 = 4 \div 4 = 1$

$\log_4 256 = \log 256 \div \log 4 = \log 4^4 \div \log 4^1 = 4 \div 1 = 4$

$\log_{16} 256 = \log 256 \div \log 16 = \log 4^4 \div \log 4^2 = 4 \div 2 = 2$

$\log_{64} 256 = \log 256 \div \log 62 = \log 4^4 \div \log 4^3 = 4 \div 3 = 1.33$

Constant P, in this case 4, divided by the term Q of both the previous terms added together, in this case 1 and 2, gives 4 ÷ (1+2) = 1.33

Part 3

If we start with $\log_a x$ and $\log_b x$ where they equal c and respectively, let us try to deduce a general statement that expresses $\log_{ab} x$ in terms of c and

We see that the base is the product of the two bases, but we only know how to multiply the arguments (where the x is), so let's transform it into:
log [base x] ab. As you know, when you reverse the base and argument, you tak e the reciprocal of the original logarithm (in this case, log [ba se x] ab = 1/log [base ab] x).

Now we can say that:
log [base x] ab = log [base x] a + log [base x] b

As I mentioned before, when the base and argument are switched you take the reciprocal, so:
log [base x] a = 1/c
log [base x] b = 1/d
log [base x] ab = 1/c + 1/d
Now let's get 1/c + 1/d into one fraction:
1/c + 1/d = d/cd + c/cd = (c + d)/cd. So:
log [base x] ab = (c + d)/cd.
We want log [base ab] x, so we want the reciprocal of that fractio n:
log [base ab] x = cd / (c + d)
Restrictions: a >0, b>0, ab ≠ 1, c ≠ -d

About the restrictions:
logs have to have positive numbers as bases, so a >0 and b>0.

Limitations:
The base must be positive and not equal to one, so:
a > 0, b > 0 (because each of them also shows up alone on a lo garithm, they can't both be negative).
Also, ab ≠ 1 -> a ≠ 1/b

Finally, the argument (x) must be positive, so:
x > 0

$$\log_a x = c \Longleftrightarrow \frac{\ln x}{\ln a} = c \ (1)$$

$$\log_b x = d \Longleftrightarrow \frac{\ln x}{\ln b} = d \ (1)$$

And we want $\log_{ab} x = \frac{\ln x}{\ln(ab)} = \frac{\ln x}{\ln a + \ln b}$

So I'm going to rewrite (1) and (2) so that I can make appear $\ln a + \ln b$

$$(1) \ \rightarrow \frac{\ln a}{\ln x} = \frac{1}{c}$$

$$(2) \ \rightarrow \frac{\ln b}{\ln x} = \frac{1}{d}$$

Therefore : $(1)+(2) \rightarrow \frac{\ln a+\ln b}{\ln x}=\frac{1}{c}+\frac{1}{d}$

Now we have found a general statement for the expression log $_{ab}x$ in terms of c and d. Where a >0, b>0, ab ≠ 1, c ≠ -d. We can therefore say that the general statement is $\frac{cd}{c+d}$.

A test of the general rule:

A = 1, B = 2, X = 2

$\log_{1\times 2} 2 = ((\log 2 \div \log 1) \times (\log 2 \div \log 2)) \div ((\log 2 \div \log 1) + (\log 2 \div \log 2))$

$1 = ((\log 2 \div 0) \times (1)) \times ((\log 2 \div 0) + (1))$

$1 = (0 \times 1) \times (0 + 1)$

$1 = \cancel{0 \times 1}$

A can not be 1.

Attachment 1;

number of logarithm	base	log		k	n	k/n
8	=2^1	=LOG(B16;C16)		3	1	=F16/G16
8	=2^2	=LOG(B17;C17)		3	2	=F17/G17
8	=2^3	=LOG(B18;C18)		3	3	=F18/G18
8	=2^4	=LOG(B19;C19)		3	4	=F19/G19
8	=2^5	=LOG(B20;C20)		3	5	=F20/G20
8	=2^6	=LOG(B21;C21)		3	6	=F21/G21
8	=2^7	=LOG(B22;C22)		3	7	=F22/G22
8	=2^8	=LOG(B23;C23)		3	8	=F23/G23
8	=2^9	=LOG(B24;C24)		3	9	=F24/G24
8	=2^10	=LOG(B25;C25)		3	10	=F25/G25

number of logarithm	base	log		k	n	k/n
25	=5^1	=LOG(B3;C3)		2	1	=F3/G3
25	=5^2	=LOG(B4;C4)		2	2	=F4/G4
25	=5^3	=LOG(B5;C5)		2	3	=F5/G5
25	=5^4	=LOG(B6;C6)		2	4	=F6/G6
25	=5^5	=LOG(B7;C7)		2	5	=F7/G7

25	=5^6	=LOG(B8;C8)		2	6	=F8/G8
25	=5^7	=LOG(B9;C9)		2	7	=F9/G9
25	=5^8	=LOG(B10;C10)		2	8	=F10/G10
25	=5^9	=LOG(B11;C11)		2	9	=F11/G11
25	=5^10	=LOG(B12;C12)		2	10	=F12/G12

Attachment 2;