Logarithmic Sequences IA

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Math SL

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In math class, I was given the assignment to evaluate multiple logarithmic sequences to see if any patterns were evident within these sequences. Using logarithmic rules that I previously learned in math class, I was able to discover multiple patterns within each logarithmic expression.

To begin, I was given the general logarithmic expression:

$$\log_m m^k, \log_{m^2} m^k, \log_{m^3} m^k, \log_{m^4} m^k, \ldots$$

In order to establish patterns within this general logarithmic expression, I will use multiple examples to help establish a common pattern between all the examples. The first sequence is as followed:

$$\log_2 8, \log_4 8 \; \log_8 8, \log_{16} 8, \log_{32} 8, \ldots$$

Next, the same logarithmic sequence will be evaluated but more in depth, to try and find a common pattern.

$$\log_2 8 = \frac{\log_2 3}{\log_2} = \frac{3\log_2}{\log_2} = \frac{3}{1}$$

$$\log_4 8 = \frac{\log_2 3}{\log_2 2} = \frac{3\log_2}{2\log_2} = \frac{3}{2}$$

$$\log_8 8 = \frac{\log_2 3}{\log_2 3} = \frac{3\log_2}{3\log_2} = \frac{3}{3}$$

$$\log_{16} 8 = \frac{\log_2 3}{\log_2 4} = \frac{3\log_2}{4\log_2} = \frac{3}{4}$$

$$\log_{32} 8 = \frac{\log_2 3}{\log_2 5} = \frac{3\log_2}{5\log_2} = \frac{3}{5}$$

Note: a pattern is already starting to show. While the numerator stays the same throughout the logarithmic sequence, the denominator increases linearly by one. The pattern is also evident within these next two examples of logarithmic sequences.

1)

$$\log_3 81 = \frac{\log_3 3^4}{\log_3 3} = \frac{4\log_3 3}{\log_3 3} = \frac{4}{1}$$

$$\log_9 81 = \frac{\log_3 3^4}{\log_3 3^2} = \frac{4\log_3 3}{2\log_3 3} = \frac{4}{2}$$

$$\log_{27} 81 = \frac{\log_3 3^4}{\log_3 3^3} = \frac{4\log_3 3}{3\log_3 3} = \frac{4}{3}$$

$$\log_{81} 81 = \frac{\log_3 3^4}{\log_3 3^4} = \frac{4\log_3 3}{4\log_3 3} = \frac{4}{4}$$

$$\log_{243} 81 = \frac{\log_3 3^4}{\log_3 3^5} = \frac{4\log_3 3}{5\log_3 3} = \frac{4}{5}$$

2)

$$\log_5 25 = \frac{\log_5 5^2}{\log_5 5} = \frac{2\log_5 5}{\log_5 5} = \frac{2}{1}$$

$$\log_{25} 25 = \frac{\log_5 5^2}{\log_5 5^2} = \frac{2\log_5 5}{2\log_5 5} = \frac{2}{2}$$

$$\log_{125} 25 = \frac{\log_5 5^2}{\log_5 5^3} = \frac{2\log_5 5}{3\log_5 5} = \frac{2}{3}$$

$$\log_{625} 25 = \frac{\log_5 5^2}{\log_5 5^4} = \frac{2\log_5 5}{4\log_5 5} = \frac{2}{4}$$

$$\log_{3125} 25 = \frac{\log_5 5^2}{\log_5 5^5} = \frac{2\log_5 5}{5\log_5 5} = \frac{2}{5}$$

Cleary, the pattern is noticeable as the sequence goes on. Therefore, I was able to find the n^{th} term for each sequence, writing it in the form $\frac{p}{q}$ where $p, q \in \mathrm{Z}$.

For the general logarithmic sequence, the n^{th} term is the following.

$$\log_{m^1} m^k = \frac{k}{1}, \log_{m^2} m^k = \frac{k}{2}, \log_{m^3} m^k = \frac{k}{3}, \log_{m^4} m^k = \frac{k}{4} \ldots, \quad \log_{m^n} m^k = \frac{k}{n}$$

That being the case, the n^{th} term for the three examples of logarithmic sequences are the following:

1) $\log_{2^n} 8 = \log_{2^n} 2^3 = \frac{3\log_2 2}{n\log_2 2} = \frac{3}{n}$

2) $\log_{3^n} 81 = \log_{3^n} 3^4 = \frac{4\log_3 3}{n\log_3 3} = \frac{4}{n}$

3) $\log_{5^n} 25 = \log_{5^n} 5^2 = \frac{2\log_5 5}{n\log_5 5} = \frac{2}{n}$

To justify my answer using technology, I used excel to verify that the pattern continues (see appendix 1). By carrying out the pattern to where n=50, I was able to confirm that the pattern continues.

Now in this investigation, I will analyze a new set of logarithmic sequences, to try and discover a new pattern that they all share in common. The new logarithmic sequences are the following:

1) $\log_4 64, \log_8 64, \log_{32} 64$

2) $\log_7 49, \log_{49} 49, \log_{343} 49$

3) $\log_4 256, \log_{16} 256 \quad \log_{64} 128$

Using the first sequence, I was able to discover that when you multiply the bases of the first and second log, you get the base of the third log, for example: 4 x 8 = 32. To evaluate the first logarithmic sequence more in depth to try and find a pattern I did the following:

$$\log_4 64 = \frac{\log 64}{\log 4} = \frac{6\log 2}{2\log 2} = \frac{6}{2}$$

$$\log_8 64 = \frac{\log 64}{\log 8} = \frac{6\log 2}{3\log 2} = \frac{6}{3}$$

$$\log_{32} 64 = \frac{\log 64}{\log 32} = \frac{6\log 2}{5\log 2} = \frac{6}{5}$$

The following equation $\frac{cd}{c+d} = x$ where *c* is the denominator of the answer on the first log, and where *d* is the denominator of the answer of the second log within the sequence. While *x* is the third term. Thus *c* = 3 and *d*= 2. To make more sense, here is the equation: $\frac{cd}{c+d} = \frac{(3)(2)}{3+2} = \frac{6}{5} = x$

To confirm that this pattern stays true, multiple other sequences will be used with the same formula: $\frac{cd}{c+d} = x$.

This is the next logarithmic sequence I will use to prove the pattern.

$\log_7 49, \log_{49} 49, \log_{343} 49$

$$\log_7 49 = \frac{\log 49}{\log 7} = \frac{\log 7^2}{\log 7^1} = \frac{2\log 7}{1\log 7} = \frac{2}{1}$$

$$\log_{49} 49 = \frac{\log 49}{\log 49} = \frac{\log 7^2}{\log 7^2} = \frac{2\log 7}{2\log 7} = \frac{2}{2}$$

$$\log_{343} 49 = \frac{\log 49}{\log 343} = \frac{\log 7^2}{\log 7^3} = \frac{2\log 7}{3\log 7} = \frac{2}{3}$$

Therefore, $\frac{cd}{c+d} = \frac{(1)(2)}{1+2} = \frac{2}{3} = x$.

Its noticeable that this pattern works for both logarithmic sequences, but to confirm that the pattern stays true, another logarithmic sequence will be used.

$\log_4 256, \log_{16} 256 \quad \log_{64} 128$

$$\log_4 256 = \log_{2^2} 2^8 = \frac{8\log_2}{2\log_2} = \frac{8}{2}$$

$$\log_{16} 256 = \log_{2^4} 2^8 = \frac{8\log_2}{4\log_2} = \frac{8}{4}$$

$$\log_{64} 256 = \log_{2^6} 2^8 = \frac{8\log_2}{6\log_2} = \frac{8}{6}$$

Therefore, $\frac{cd}{c+d} = \frac{(2)(4)}{2+4} = \frac{8}{6} = x$

This pattern seems to work for the above examples of logarithmic sequences, but to prove that this pattern continues to work for all logarithmic sequences like the ones above then a general formula must be established.

If the following two statements are true,

$$\log_a x = C$$

$$\log_b x = D$$

Then this next statement must also be true,

$$\log_{ab} x = \frac{CD}{C+D}$$

Therefore, the statements above will be used to prove that $\frac{cd}{c+d} = x$ within this next statement:

$$\log_{ab} x = \frac{\log_a x}{\log_a ab} = \frac{C}{\log_a a + \log_a b} = \frac{C}{1+\log_a b}$$
$$\log_{ab} x = \frac{\log_b x}{\log_b ab} = \frac{D}{\log_b a + \log_b b} = \frac{D}{\log_b a + 1}$$
$$\frac{D}{\log_b a + 1} = \frac{C}{1+\log_a b}$$

This investigation aimed to prove that this equation:

$$\log_{ab} x = \frac{CD}{C+D}$$

is true for all similar logarithmic sequences. How ever, this cannot be used for any log where the base of *a* times the base of *b* does not equal *c*. Also this equation does not work for negative numbers because you cannot take the log of a negative number.

In conclusion, I arrived at my general statement for both parts by using multiple logarithmic sequences to confirm that a pattern actually occurs, and is true for all similar logarithmic sequence.

Appendix 1: shows the continuation of pattern within Part one of IA.

index	base	log(base)8
1	2	3
2	4	1.5
3	8	1
4	16	0.75
5	32	0.6
6	64	0.5
7	128	0.428571429
8	256	0.375
9	512	0.333333333
10	1024	0.3
11	2048	0.272727273
12	4096	0.25
13	8192	0.230769231
14	16384	0.214285714
15	32768	0.2
16	65536	0.1875
17	131072	0.176470588
18	262144	0.166666667
19	524288	0.157894737
20	1048576	0.15
21	2097152	0.142857143
22	4194304	0.136363636
23	8388608	0.130434783
24	16777216	0.125
25	33554432	0.12
26	67108864	0.115384615
27	134217728	0.111111111
28	268435456	0.107142857
29	536870912	0.103448276
30	1073741824	0.1
31	2147483648	0.096774194
32	4294967296	0.09375
33	8589934592	0.090909091
34	17179869184	0.088235294
35	34359738368	0.085714286
36	68719476736	0.083333333
37	1.37439E+11	0.081081081

38	2.74878E+11	0.078947368
39	5.49756E+11	0.076923077
40	1.09951E+12	0.075
41	2.19902E+12	0.073170732
42	4.39805E+12	0.071428571
43	8.79609E+12	0.069767442
44	1.75922E+13	0.068181818
45	3.51844E+13	0.066666667
46	7.03687E+13	0.065217391
47	1.40737E+14	0.063829787
48	2.81475E+14	0.0625