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# Mathematics Internal Assessment

## LOGARITHM BASES

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IB Mathematics-SL

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# Part- 1

3 sequences and their expression in the $m^{th}$ term has been given. All of these equations will be evaluated on a step-by-step basis in order to find an expression for the $n^{th}$ term. The technology which will be used to execute this investigation will be a TI-83 graphing calculator and Logger Pro graphing Program. All mathematical calculations have been made by this graphing calculator and is mentioned where some calculations have not been shown.

## (A)

Let us observe the first row of the sequence.

The following sequence is given as:

$\log_2 8, \log_4 8, \log_8 8, \log_{16} 8, \ldots$

Each of the terms in this sequence are of the form $\log_b A$. If analyzed, it can be easily observed that the log base $(b)$ of each term in this sequence is a power of 2 while $A$ remains constant at 8.

Therefore, we can state that each term for the $1^{st}$ row has the general expression $\log_{2^n} 8$, where n is the $n^{th}$ term of the sequence. Now, we must apply this expression in the form $\frac{p}{q}$.

$\log_{2^n} 8$

$= \frac{\log 8}{n \log 2}$ [Applying change of base formula]

$= \frac{2 \log 3}{n \log 2}$ [Substituting 8 as a power of 2]

$= \frac{3}{n}$

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Now, the expression $\frac{3}{n}$ , should provide us with the values of each term in the sequence. Thus, to prove $\frac{3}{n}$ as the general expression for the first row of sequences:

$U_n = \frac{3}{n}$

First term of the sequence = $\log_2 8$

$\frac{3}{n} = \log_{2^n} 8$

(Taking the first term as and so on,)

| | |
|---|---|
| L.H.S | R.H.S |
| $= \frac{3}{1}$ | $\log_{2^1} 8$ |
| $U_1 = 3$ | $\log_2 8 = 3$ |

L.H.S = R.H.S. Hence, verified that $\frac{3}{n}$ is the general expression for the $1^{st}$ row of the given sequence in the form $\frac{p}{q}$ .

Similarly, other terms in the sequence were also verified by using the TI-83 where:

$U_2$: $\frac{3}{2} = \log_4 8$ (1.5)

$U_3$: $\frac{3}{3} = \log_8 8$ (1)

$U_4$: : $\frac{3}{4} = \log_{16} 8$ (0.75)

$U_5$: : $\frac{3}{5} = \log_{32} 8$ (0.6)

$U_6$: : $\frac{3}{6} = \log_{64} 8$ (0.5)

$U_7$ : : $\frac{3}{7} = \log_{128} 8$ (0.42)

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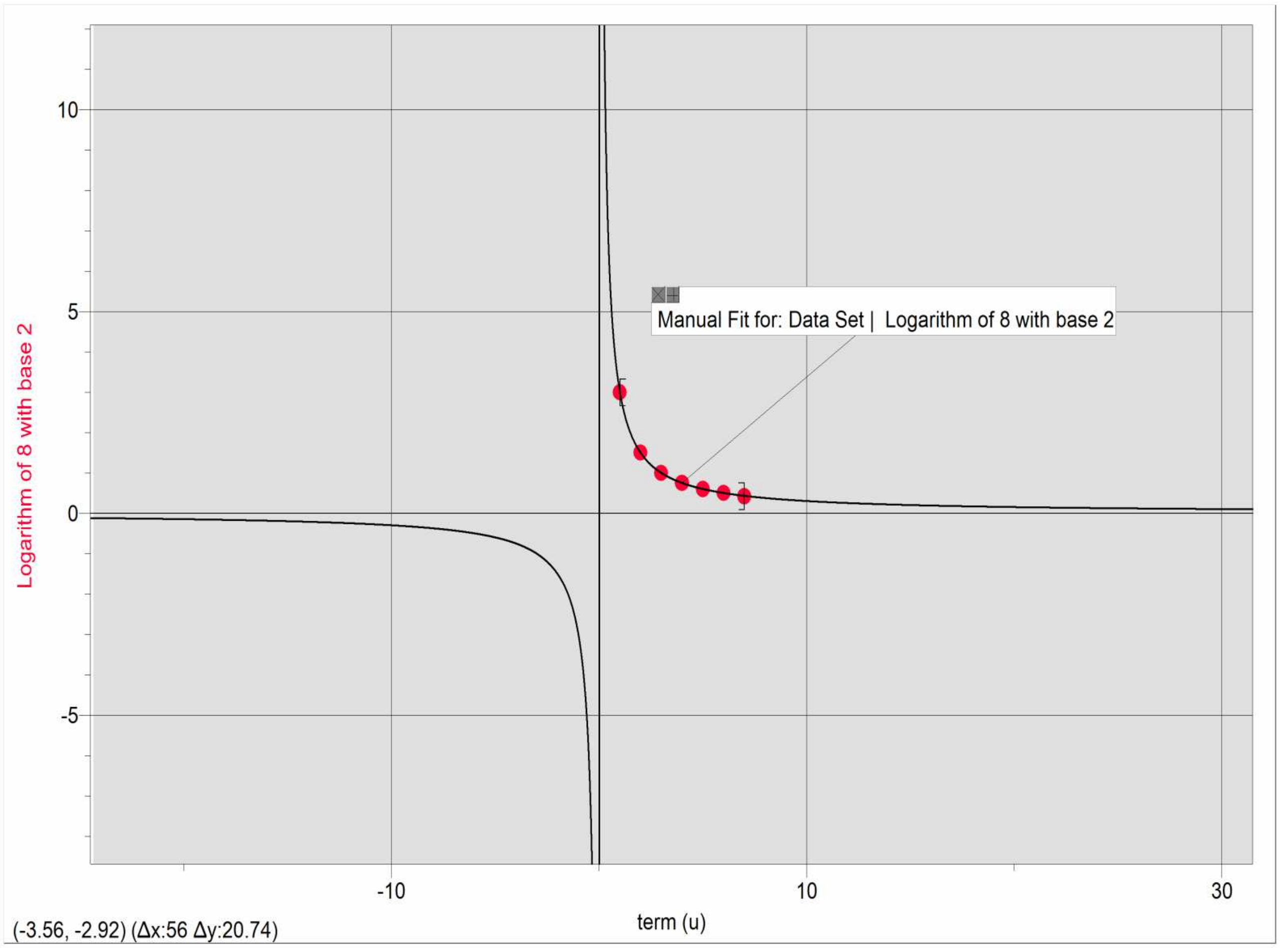
Below is the graph for the reciprocal function f(x)= $\frac{3}{x}$




Now, below is the Graph emphasizing values of given terms of the sequence.

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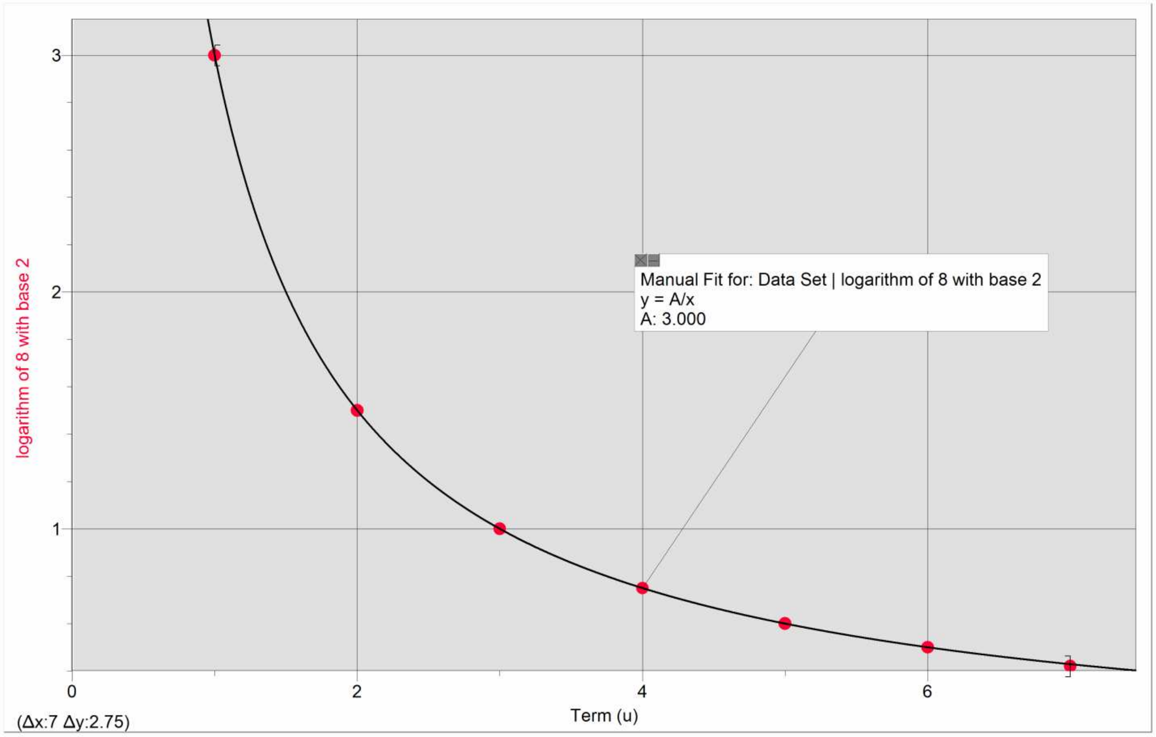


The graph above is a result of plotting f(x)= $\frac{3}{x}$ and plotting the values of each term in the sequence [x | x≥0] and [y | y≥0]. For our purposes, this graph is confined to the positives as the points in red( which represent the values of the respective terms in the sequence) are positive values. From the graph, we can observe that the y-value of the co-ordinates plotted. These are in fact, the values of logarithmic terms while the x value is the term of the sequence. For example,

To find that value of logarithmic term $u_{4}= \log_{16} 8$,

Through applying the change of base formula in the Graphing Calculator we get,

$\frac{log8}{log16}$ = 0.75

(4,0.75) is plotted on the plane and it is also point on the graph, f(x) = $\frac{3}{x}$. Thus, in order for a logarithm to be a part of the sequence of the first row, its value along

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with the term number() (where the value is a y co-ordinate and () is the x-coordinate) must be a point on the graph f(x) = $\frac{3}{x}$.

Thus, the next two terms of the sequence have been verified both algebraically and graphically.

# (B)

The sequence for the second row is given as:

$$\log_3 81, \log_9 81, \log_{27} 81, \log_{81} 81, \ldots$$

Similarly , the sequence of the 2nd row has the expression:

$\log_{3^n} 81$ (this is because, the bases (b) of each of the terms in the sequence are powers of 3 while , which is 81, remains the same for each term)

$= \frac{\log 81}{n \log 3}$ [Applying change of base formula]

$= \frac{4 \log 3}{n \log 3}$ [Substituting 81 as a power of 3]

$= \frac{4}{n}$

Now, the expression $\frac{4}{n}$ , should provide us with the values of each term in the sequence. Thus, to prove $\frac{4}{n}$ as the expression for the second row of sequences in the form $\frac{p}{q}$ :

$U_n = \frac{4}{n}$

First term of the sequence = $\log_3 81$

$\frac{4}{n} = \log_{3^n} 81$

(Taking the first term as and so on,)

L.H.S R.H.S

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$= \frac{4}{1}$ $\log_{3^1} 81$

$U_1 = 4$ $\log_3 81 = 4$

L.H.S = R.H.S. Hence, it is verified that $\frac{4}{n}$ is the general expression for the 2nd row of the given sequence in the form $\frac{p}{q}$.

Similarly, other terms in the sequence were also verified by using the TI-83 where:

$U_2$: $\frac{4}{2} = \log_9 81$ (2)

$U_3$: $\frac{4}{3} = \log_{27} 81$ (1.33)

$U_4$: : $\frac{4}{4} = \log_{81} 81$ (1)

$U_5$: : $\frac{4}{5} = \log_{243} 81$ (0.8)

$U_6$: : $\frac{4}{6} = \log_{729} 81$ (0.66)

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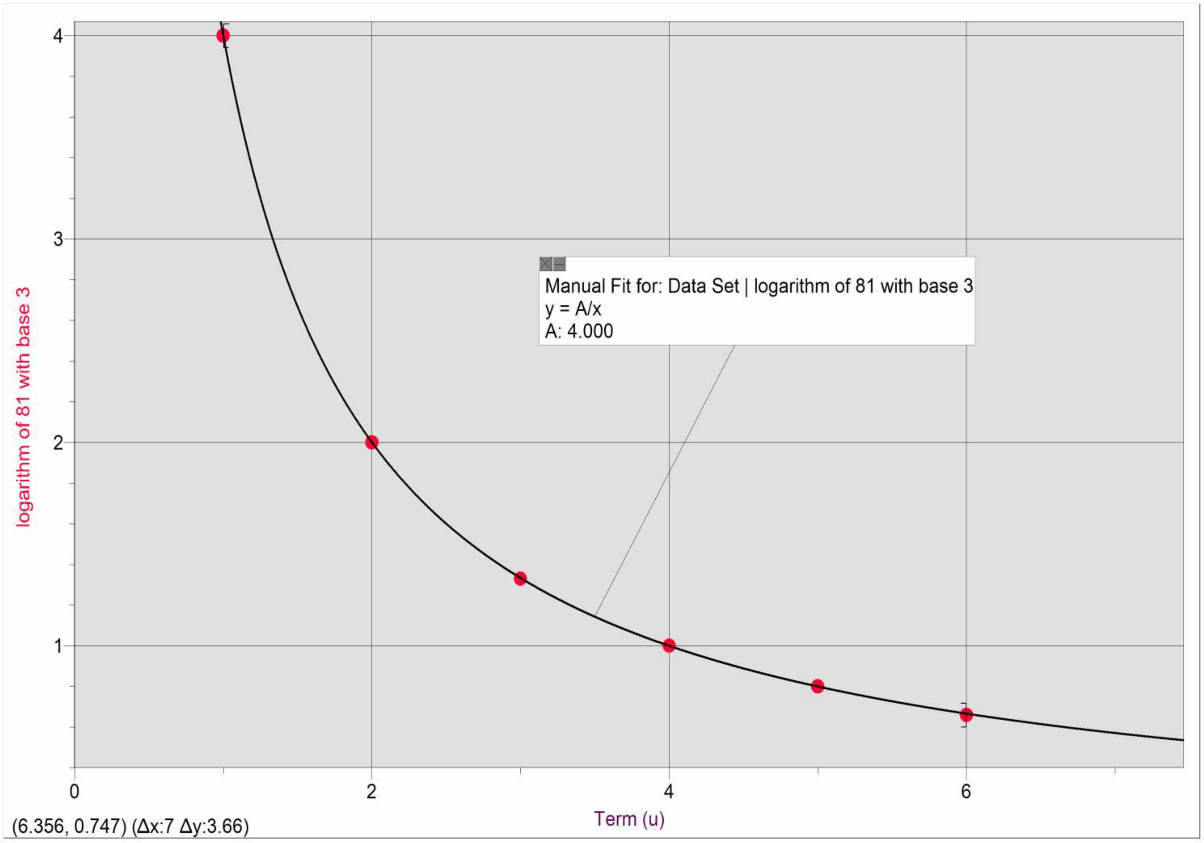


The graph above is a result of plotting f(x)= $\frac{4}{x}$ and plotting the values of each term in the sequence.

## (C)

Now, the sequence of the 3rd row has the expression:

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$\mathrm{Log}_{5^n} 25$ (this is because, the bases (b) of each of the terms in the sequence are powers of 5 while which is 25, remains the same for each term)

$= \frac{\log 25}{n \log 5}$ [Applying change of base formula]

$= \frac{2 \log 5}{n \log 5}$ [Substituting 25 as a power of 5]

$= \frac{2}{n}$

Now, the expression $\frac{2}{n}$, should provide us with the values of each term in the sequence. Thus, to prove $\frac{2}{n}$ as the expression for the second row of sequences in the form $\frac{p}{q}$:

$U_n = \frac{2}{n}$

First term of the sequence = $\log_5 25$

$\frac{2}{n} = \log_{5^n} 25$

(Taking the first term as and so on,)

L.H.S R.H.S

$= \frac{2}{1}$ $\log_{5^1} 25$

$U_1 = 2$ $\log_5 25 = 2$

Similarly, other terms in the sequence were also verified by using the TI-83 where:

$U_2$: $\frac{2}{2} = \log_{25} 25$ (1)

$U_3$: $\frac{2}{3} = \log_{125} 25$ (0.66)

$U_4$: : $\frac{2}{4} = \log_{625} 25$ (0.5)

$U_5$: : $\frac{2}{5} = \log_{3125} 25$ (0.4)

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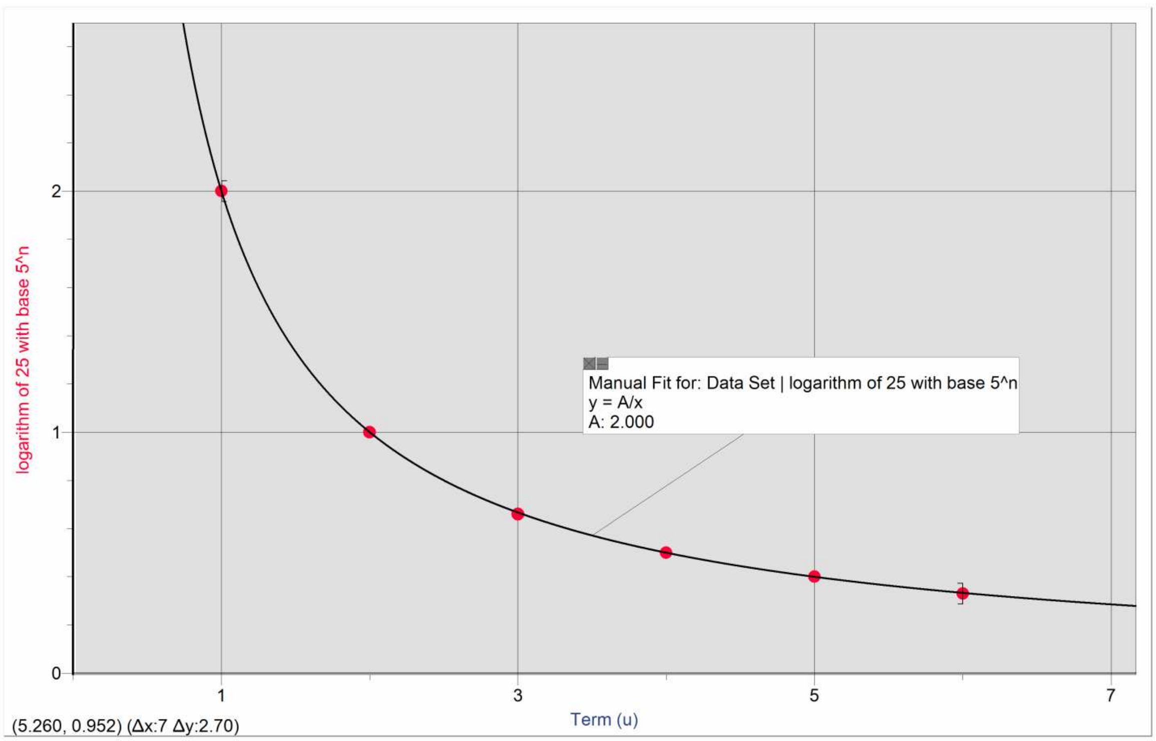
$U_6$: : $\frac{2}{6} = \log_{15625} 25$ (0.33)

L.H.S = R.H.S. Hence, Verified that $\frac{2}{n}$ is the general expression for the 3rd row of the given sequence in the form $\frac{p}{q}$.




# (D)

Also given, is the sequence for the $m^{th}$ term:

$\text{Log}_m$ [illegible]$^k$, $\log_{m^2}$ [illegible]$^k$ , $\log_{m^3}$ [illegible]$^k$, $\log_{m^4}$ [illegible]$^k$, …

Thus, we can establish the expression of this sequence to be:

$\text{Log}_{m^n} m^k$

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$= \frac{k \log m}{n \log m}$

$= \frac{k}{n}$

Therefore, the general expression $\frac{k}{n}$ (which is in the form $\frac{p}{q}$ ) can be applied to all the given sequences. We can apply this general expression to verify the successive terms of each sequence.

| Term ( ) | Logarithm of 8 with base $2^n$ | Logarithm of 81 with base $3^n$ | Logarithm of 25 with base $5^n$ |
|---|---|---|---|
| 1 | 3.00 | 4.00 | 2.00 |
| 2 | 1.50 | 2.00 | 1.00 |
| 3 | 1.00 | 1.33 | 0.66 |
| 4 | 0.75 | 1.00 | 0.50 |
| 5 | 0.60 | 0.8 | 0.40 |
| 6 | 0.50 | 0.66 | 0.33 |
| 7 | 0.42 | - | - |

Below is the general graph for the functions f(x)= $\frac{3}{x}$ , g(x) = $\frac{4}{x}$ , h(x) = $\frac{2}{x}$

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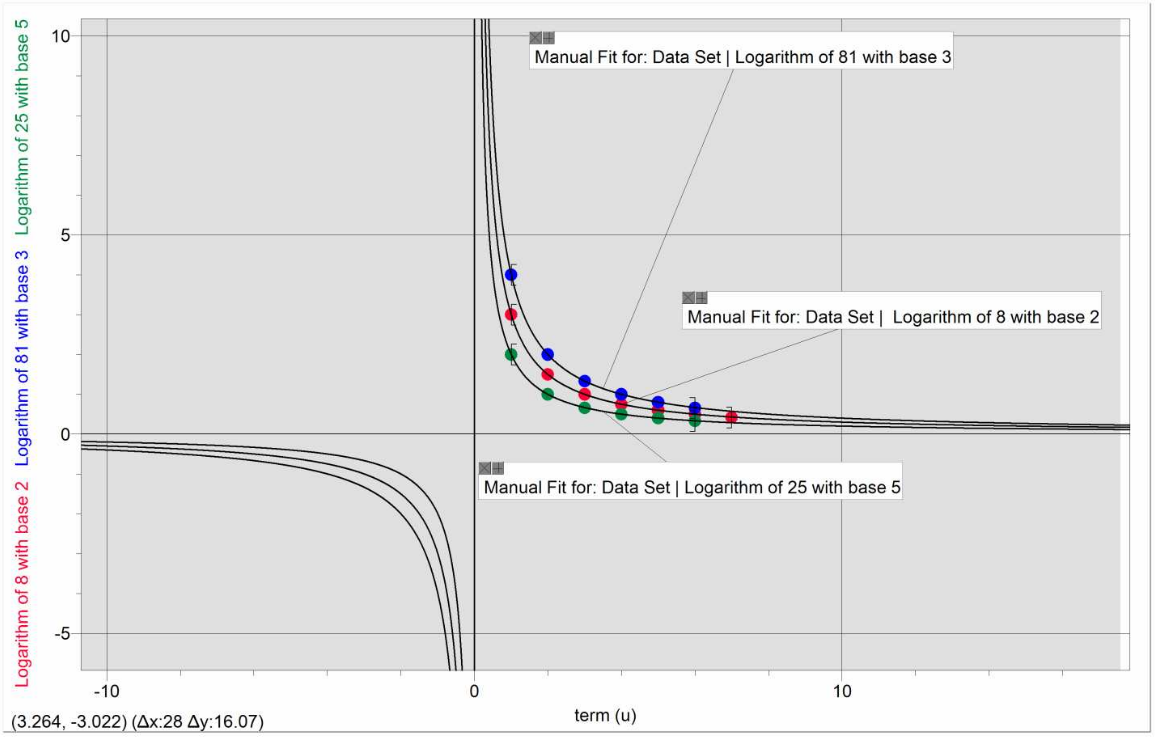


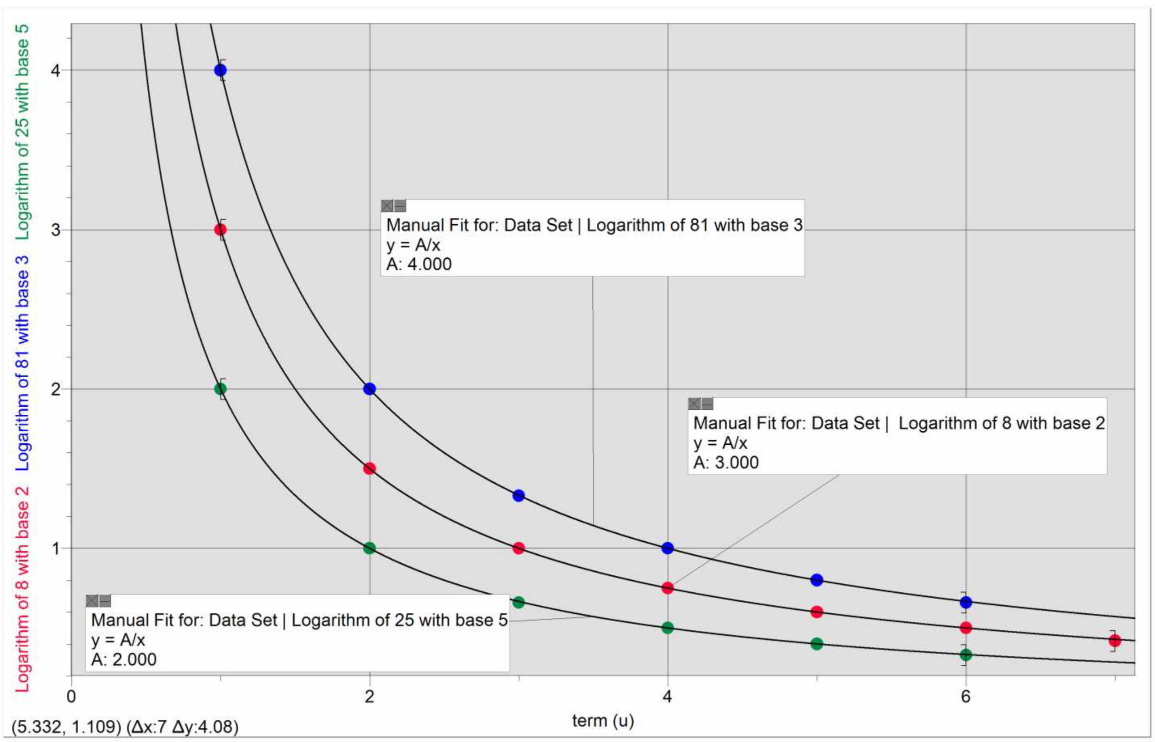

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The above graph shows all three expressions from the given sequences plotted together with the values of each term in their respective sequence. Now, if we observe the graph above very carefully, we can see that the points are on a straight line, but most importantly, we can also observe that each point is equidistant from the next for each term. If this is proved to be true, one can graph even more functions to add to the given set of sequences.

To prove, the distance between points [illegible] and a, and a and c is the same for each term on the x axis.

That is,   C-A= B-C

Now, by extracting the values from the same table given above earlier,  we can solve the mentioned problem .

| Term ( $n$ ) | Logarithm of 8 with base $2^n$ (c) | Logarithm of 81 with base $3^n$ (b) | Logarithm of 25 with base $5^n$ (a) |
|---|---|---|---|
| 1 | 3.00 | 4.00 | 2.00 |
| 2 | 1.50 | 2.00 | 1.00 |
| 3 | 1.00 | 1.33 | 0.66 |
| 4 | 0.75 | 1.00 | 0.50 |
| 5 | 0.60 | 0.80 | 0.40 |
| 6 | 0.50 | 0.67 | 0.33 |
| 7 | 0.42 | - | - |

**Verification : C-A = B-C**

[Calculations]

Term 1:  3-2 = 4-3  = 1

Term 2:  1.5-1.0=2.0-1.5= 0.50

Term 3:  1.0-0.7 (rounded from 0.66)=1.3-1.0 = 0.30

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Term4: 0.75-0.50= 1.00-0.75 = 0.25

Term 5: 0.60-0.40=0.80-0.60 = 0.20

Term 6: 0.50-0.33= 0.67-0.50 = 0.17

| Term | C-A | B-C | Common Difference (d) |
|---|---|---|---|
| 1 | 3-2 | 4-3 | 1 |
| 2 | 1.5-1.0 | 2.0-1.5 | 0.50 |
| 3 | 1.0-0.7 (rounded from 0.66)= | 1.3-1.0 | 0.30 |
| 4 | 0.75-0.50 | 1.00-0.75 | 0.25 |
| 5 | 0.60-0.40 | 0.80-0.60 | 0.20 |
| 6 | 0.50-0.33 | 0.67-0.50 | 0.17 |

| Logarithm of 81 with base $3^n$ (b) | New points on the plane above $y = \frac{4}{x}$. |
|---|---|
| 4.00 | 5.00 |
| 2.00 | 2.50 |
| 1.33 | 1.63 |
| 1.00 | 1.25 |
| 0.80 | 1.00 |
| 0.67 | 0.84 |

Hence proved, that each point is equidistant to the next for each point on the x-axis (the term of the sequence). Now that the common difference has been found, we can plot more points on the graph according to the difference in height. Below is a graph with differences by adding the differences graphically, starting from the topmost graph $y = \frac{4}{x}$.

| Term ( $x$ | Logarithm of 8 with base $2^n$ (c) | Logarithm of 81 with base $3^n$ (b) | Logarithm of 25 with base $5^n$ (a) | New points on the plane above $y = \frac{4}{x}$. |
|---|---|---|---|---|
| 1 | 3.00 | 4.00 | 2.00 | 5.00 |
| 2 | 1.50 | 2.00 | 1.00 | 2.50 |
| 3 | 1.00 | 1.33 | 0.66 | 1.63 |
| 4 | 0.75 | 1.00 | 0.50 | 1.25 |
| 5 | 0.60 | 0.80 | 0.40 | 1.00 |
| 6 | 0.50 | 0.67 | 0.33 | 0.84 |

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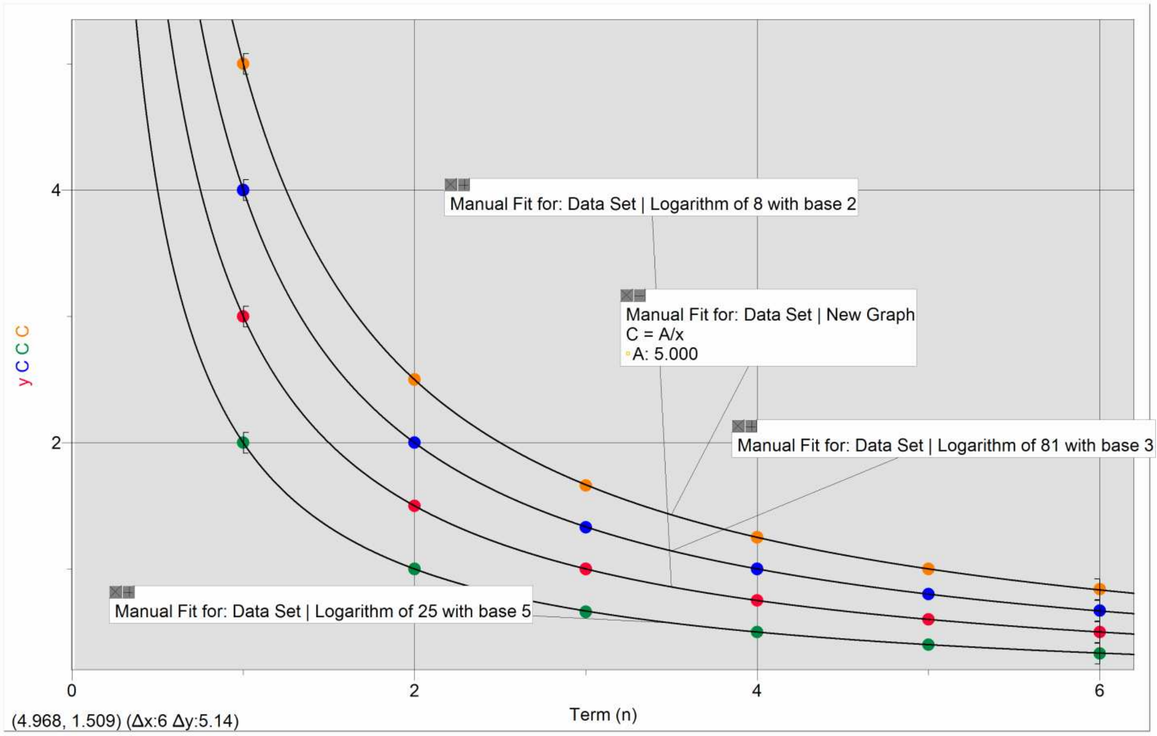


**Full view of graph**

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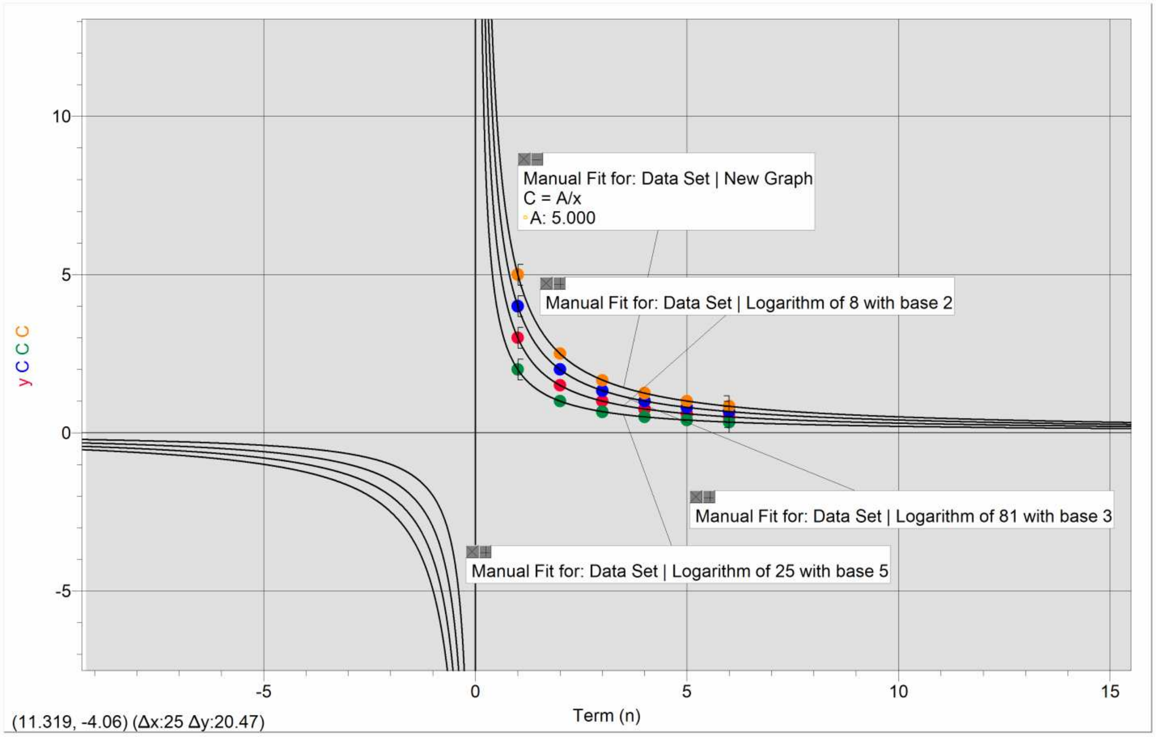


Thus, as seen in the graphing program, a new graph has been created on the bases of the common difference found between the distances between the points of the previous graphs. The new graph has the equation of y= $\frac{5}{x}$ in the form $\frac{p}{q}$.

# PART 2.

All calculations have been made by the TI-83 graphing calculator and this has been mentioned where some calculations have not been shown. The aim is to obtain two more examples fitting the pattern of the sequence given below, in addition to describing how to obtain the third answer in each row from the first two answers.

The given sequence is

$\text{Log}_4 64$, $\log_6 64$, $\log_{32} 64$

$\text{Log}_7 49$, $\log_{49} 49$, $\log_{343} 49$

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Log$_{\frac{1}{5}}$ 125, log$_{\frac{1}{125}}$ 125, log$_{\frac{1}{625}}$ 125

Log$_8$ 512, log$_2$ 512, log$_{16}$ 512

# (A)

The terms in the sequence above are in the form of $\log_b A$

Now, in order to obtain the third answer in each row from the first two answers, one must multiply the bases of the first two terms in the sequence to obtain the base of the third term. For example, in the first sequence, we multiply the bases 4 and 6 to obtain 32 while the logarithm of A remains constant for that sequence. Thus making the third term log$_{32}$ 64. This is true for all given sequences and has been verified with the use of the TI-83 calculator.

Therefore, if U$_1$= $\log_a x$

And U$_2$ = $\log_b x$

Then U$_3$= $\log_{ab} x$

If $\log_a x$ = c

And, $\log_b x = d$;

In terms of c and d we can express the value of $\log_{ab} x$ as $\frac{cd}{c+d}$

To prove value of $\log_{ab} x$ = $\frac{cd}{c+d}$

In the second row we have, log$_7$ 49 and log$_{49}$ 49, where c=2 and d= 1. And a=7 and b=49

The value of the third term ($\log_{ab} x$) is $\frac{2 \times 1}{2+1}$ = $\frac{2}{3}$

To verify if the value of the third term $\log_{343} 49 = \frac{2}{3}$

Hence, by evaluating the third term through the graphing calculator while using the change of base formula, it is confirmed that the value of $\log_{343} 49 = \frac{2}{3}$.

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Similarly, all the evaluated values of last term in each sequence were the same as the values of the first two terms when they were evaluated with the formula $\frac{cd}{c+d}$ with the help of the graphing calculator.

## (B)

Let us take the pattern above while regarding base $\frac{1}{5}$ as 5 from the bases of the first column. We get:

4

7

5

8

We see that there is a difference of 3 between the 2 pairs of terms. We observe that after adding 3 to $U_1$ in order to get $U_2$, we subtract 2 from $U_2$ to get $U_3$ and so on. Therefore, the next two terms of this pattern will be 6 and 9.

Now, lets consider the pattern of the second column. This time, not the base of the logarithm, but the value of the logarithm will be taken. Thus for the second column after calculating the values of the terms, we obtain the values: 3 ($\log_8 64$), 1 ($\log_{49} 49$), -1 ($\text{Log}_{\frac{1}{125}} 125$) and 9 ($\log_2 512$) respectively.

Similarly, let us get the values of the logarithms in the first column. We obtain 3($\log_4 64$), 2($\log_7 49$),-3($\text{Log}_{\frac{1}{25}} 125$) and 3 ($\log_8 512$) .

Now, we have found out the all values of c and d for in the sequences given to us. The grid looks like:

| Column 1(c) | Column 2(d) |
|---|---|
| 3 | 2 |
| 2 | 1 |
| -3 | -1 |
| 3 | 9 |

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However, we can now get back to our formula $\frac{cd}{c+d}$, we can now add values to the third column by applying this formula to each of the values in the first and second columns respectively .(with the help of the graphing calculator)

Therefore, Column 3

1.5

0.66

-.075

-4.5

Thus, by verifying with a graphing calculator, these values above are equal to the values of the logarithmic terms given in the sequences.

Therefore, following the pattern, another example which will fit the pattern above is.

$\text{Log}_6\ 36$, $\log_{36} 36$, $\log_{216} 36$

$\text{Log}_{\frac{1}{9}}729$, $\log_{729} 729$, $\log_{81} 729$

This is proved by :

$\text{Log}_6\ 36= 2$

$\log_{36} 36= 1$

$\log_{216} 36= \frac{2}{3}$

according to the formula $\frac{cd}{c+d}$, we get $\frac{2\times 1}{2+1} = \frac{2}{3}$

Hence, since $\frac{cd}{c+d}$ is equal to the value of the last term, the first sequence is verified.

For the second sequence,

We obtain $\text{Log}_{\frac{1}{9}}729 = -3$

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$\log_{729} 729 = 1$

$\log_{81} 729 = 1.5$

according to the formula $\frac{cd}{c+d}$, we get $\frac{-3\times1}{(-3)+1} = \frac{-3}{-2} = \frac{3}{2}$ or 1.5

Hence, since $\frac{cd}{c+d}$ is equal to the value of the last term, the second sequence is also verified.

The limitations of a, b and x is that we can only take positive values of a and b because logarithms of negative values don't exist. Therfore, a, b and x can't be negatives even though it is given that p,q, $\in Z$. Therfore, p,q are only confined to p,q[$\in Z^+$]