The Math that is not shown is done by a GDC.
I used the Texas Instruments TI – 83 Plus.

In this analysis of logarithm bases I will be discussing and explaining some of the properties and functions dependent on formulas which can be obtained by the use of logarithms. Simply, a logarithm is just an exponent. View the following diagram below in figure 1. An alternate explanation is that, the logarithm of a number x to a base b is just the exponent you put onto b to make the result equal x. For instance, since $5^2 = 25$, we know that 2 (the exponent to the five) is the logarithm of 25 to base 5. Represented as, $\log_5(25) = 2$.

If given the base 2, for example, and exponent 3, then we can evaluate 2^3.

$2^3 = 8$.

How about if we are given the base 2 and its power 8 --

$2^? = 8$

$2^? = 2^3$

$? = 3$

-- then what is the exponent that will produce 8?

That exponent is called a logarithm. Resulting as the exponent that was 3 is now the logarithm of 8 with base 2. Written as,

$3 = \log_2 8$.

We write the base 2 as a subscript. Hence, not stir up confusion.

3 is the exponent to which 2 must be raised to produce 8.

Therefore, a logarithm is simply an exponent.

Figure 1.

In a general expression, when applying variables to replace numbers, if $x = b^y$, then we say that y is "the logarithm of x to the base b" or "the base b logarithm of x". Represented as, $y = \log_b(x)$. Therefore, any exponential equation can be written as a log just by changing the x and y positions to manipulate the equation.

Another way to look at it is that the $\log_b x$ function is defined as the inverse of the b^x function. These two statements express that inverse relationship, showing how an exponential equation is equivalent to a logarithmic equation. View the diagrams in figure 2 and figure 3.

$\boldsymbol{x = b^y}$ is the same as $\boldsymbol{y = \log_b x}$

Exponential. Logarithmic

$1000 = 10^3$ is the same as $3 = \log_{10} 1000$.

Figure 2. **Figure 3.**

Logarithms follow exponential rules being exponents themselves. A few general statements include that one can observe in any base, the logarithm of 1 is 0. View the diagram in figure 4. Up until this point the examples used, contained positive numbers. Can we find the logarithm of a negative number? View the diagram in figure 5. In addition, in any base, the logarithm of the base itself is 1. View the diagram in figure 6. Logarithms follow other rules such as inverse properties. View the diagram in figure 7.

Calculate $\log_8 1$.
8 to what exponent produces 1? $8^0 = 1$.

$\log_8 1 = 0$.

General expression: $\log_b 1 = 0$
$b > 1$
$b \neq 1$ because $1 = 1^b$
This will always equal 1 for any value of b.

Figure 4.

To calculate $\log_a(-4)$ for some base $a > 0$
It has to be understood that:
$X = \log_a(-4)$ manipulated to $a^x = -4$
However, $a^X = -4$ where $a > 0$, will always be positive, hence, there is no value for $a^X = -4$.
This means there cannot be a "real" solution to a logarithm of a negative number. However, an imaginary one can be found.

*The remainder of this portfolio will not deal with negative logarithms. Unless to show restrictions.

Figure 5.

Calculate $\log_5 5$.

5 with what exponent will produce 5? $5^1 = 5$.

$\log_5 5 = 1$.

In any base, the logarithm of the base itself is 1.

General expression: $\log_b b = 1$

Figure 6.

Inverse properties: $\log_B B^x = x$ and $B^{(\log_B x)} = x$

Ex.) 2 raised to what exponent will produce 2^m ? m, obviously.

$\log_2 2^m = m$.

Figure 7.

Can you take the logarithm of square roots? View figure 8 below.

$\log_3 \sqrt[5]{3} = ?$

$\sqrt[5]{3} = 3^{1/5}$. Therefore,

$\log_3 \sqrt[5]{3} = \log_3 3^{1/5} = 1/5$.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

Change to rational roots.

Figure 8.

Now, there are two basic logarithms common and natural. Beginning with common, the logarithm base 10 of a number. That is, the power of 10 necessary to equal a given number. The common logarithm of x is written log x. For example, log 100 is 2 since $10^2 = 100$. The base is understood to be ten. View the diagram in figure 9. The natural logarithm is composed of the logarithm base e of a number. The system of natural logarithms has the number called "e" as its base. Why *e?* (e is named after the 18th century Swiss mathematician, Leonhard Euler.) It is called the "natural" base because of certain technical considerations. That is, the power of e necessary to equal a given number. The natural logarithm of x is written ln x. For example, ln 8 is 2.0794415... since $e^{2.0794415...} = 8$. How is e calculated? View the diagram in figure 10. ln is the symbol for natural log.

The system of common logarithms has 10 as its base. When the base is not indicated,

log 100 = 2 $10^2 = 100$.

then for the system of common logarithms -- base 10 -- is implied.

Here are the powers of 10 and their logarithms for a few examples:

Powers of 10:	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1000	10,000
Logarithms:	−3	−2	−1	0	1	2	3	4

Figure 9.

e can be calculated from the following series involving factorials:

A factorial is when you multiply by consecutive numbers following n!

$n! = 1\cdot 2\cdot 3\cdot\cdot\cdot(n-2)(n-1)n$

ex.) $6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 720$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ . \ . \ .$$

e is an irrational number, whose decimal value is approximately

2.71828182845904.

To indicate the natural logarithm of a number, we use the notation "ln." ln x means $\log_e x$.

Figure 10.

There are three laws of logarithms:

1.) "*The logarithm of a product is equal to the sum of the logarithms of each factor.*"
   a. $\log_b xy = \log_b x + \log_b y$
2.) "*The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.*"
   a. $\log_b \frac{x}{y} = \log_b x - \log_b y$
3.) "*The logarithm of a power of x is equal to the exponent of that power times the logarithm of x.*"
   a. $\log_b x^n = n \log_b x$

The laws of logarithms will be valid for any base. I will prove this using base *e*, that is, for $y = \ln x$. I am going to prove law number 1. "*The logarithm of a product is equal to the sum of the logarithms of each factor.*"

View the diagram in figure 11, below.

$$\ln ab = \ln a + \ln b.$$

The function $y = \ln x$ is defined for all positive real numbers x. Therefore there are real numbers p and q such that

$p = \ln a$ and $q = \ln b$.

This implies

$a = e^p$ and $b = e^q$.

Therefore, according to the rules of exponents,

$ab = e^p \cdot e^q = e^{p+q}$.

And therefore

$\ln ab = \ln e^{p+q} = p + q = \ln a + \ln b$

Careful!

$\log_a (x + y) \neq \log_a x + \log_a y$

$\log_a (x - y) \neq \log_a x - \log_a y$

Figure 11.

The second law can be proved in the same way. Instead of addition in the steps it would be subtraction. View the example in Figure 12 to see an example of the first three laws in a problem.

$$\log \frac{abc^2}{d^3} = \log(abc^2) - \log d^3 = \log a + \log b + \log c^2 - \log d^3$$
$$= \log a + \log b + 2\log c - 3\log d.$$

Figure 12.

I will prove the third law, "*The logarithm of a power of x is equal to the exponent of that power times the logarithm of x.*" View Figure 13, below.

$$\ln a^n = n \ln a.$$

There is a real number p such that $p = \ln a$; that is, $a = e^p$.

And the rules of exponents are valid for all rational numbers n. An irrational number is the limit of a sequence of rational numbers. Therefore,

$a^n = e^{pn}$. This implies $\ln a^n = \ln e^{pn} = pn = np = n \ln a$.

As seen in Figure 12 $\log d^3 = 3 \log d$.

Figure 13.

As you know not all logs will hold a natural or common logarithm base such as 10 or e. Therefore, change of base is necessary. View the diagram in Figure 14.

You know the value of logarithms of base 10, but not, for example, in base 2. Then you can convert a logarithm in base 10 to one in base 2 -- or any other base -- by realizing that the values will be proportional.

$$\log_2 x = k \log x.$$

Each value in base 2 will differ from the value in base 10 by the same constant k.

Now, to find that constant, you know that $\log_2 2 = 1.$

Therefore, on putting $x = 2$: $\log_2 2 = k \log 2 = 1,$

which implies $k = \dfrac{1}{\log 2}.$

Therefore, $\log_2 x = \dfrac{\log x}{\log 2}.$

By knowing the values in base 10, you can in this way calculate the values in base 2.

In general, if known the values in base a, then you can change to base b as follows: $\log_b x = \dfrac{\log_a x}{\log_a b}.$

Figure 14.

What is another example of the change of base? View figure 15.

Convert $log_3(6)$ to base 5.

$$log_3(6) = \frac{log_5(6)}{log_5(3)}$$

Figure 15.

base of logarithms	symbol	name
10	log (if no base shown)	common logarithm
e	ln	natural logarithm, pronounced "ell-enn" or "lahn"

The "base"ics

From the definition of a log as inverse of an exponential, you can immediately get some basic facts. For instance, if you graph $y=10^x$ (or the exponential with any other positive base), you see that its range is positive reals; therefore the domain of $y=\log x$ (to any base) is the positive reals. In other words, you can't take log 0 or log of a negative number unless you want to deal with imaginary numbers. View the diagram in figure 16 to see the graph of $y=10^x$.

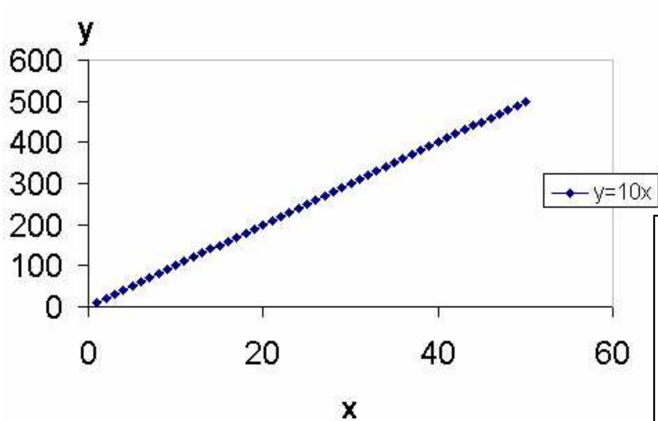

Figure 16.

A less congested version of all the rules and a few more can be viewed in the table in figure 17, below.

Useful applications: DON'T FORGET!

log $(5+x)$ is not the same as log 5 + log x. As you know, log 5 + log x = log$(5x)$, not log$(5+x)$. Look carefully at the table and you'll see that there's nothing you can do to split up log$(x+y)$ or log$(x-y)$.

(log x) / (log y) is not the same as log(x/y). In fact, when you divide two logs to the same base, you're working the change-of-base formula backward. Though it's not often useful, (log x) / (log y) = $\log_y x$. Just don't write log(x/y)!

(log 5)(log x) is not the same as log$(5x)$. You know that log$(5x)$ is log 5 + log x. There's really not much you can do with the product of two logs when they have the same base.

exponents	logarithms
(All laws apply for any positive a, b, x, and y.)	
$x = b^y$ is the same as $y = \log_b x$	
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
$b^{(\log_b x)} = x$	$\log_b b^x = x$
$b^x\, b^y = b^{x+y}$	$\log_b(xy) = \log_b x + \log_b y$
$b^x \div b^y = b^{x-y}$	$\log_b(x/y) = \log_b x - \log_b y$
$(b^x)^y = b^{xy}$	$\log_b(x^y) = y \log_b x$
	$(\log_a b)(\log_b x) = \log_a x$
	$\log_b x = (\log_a x) / (\log_a b)$
	$\log_b a = 1 / (\log_a b)$

Figure 17.

There you have it! The general idea of what are logarithms. The puzzle pieces have been put together. Now onto the exploration of logarithm bases!

Problem 1:
Consider the following sequences. Write down the next two terms of each sequence.

- $\log_2 8$, $\log_4 8$, $\log_8 8$, $\log_{16} 8$, $\log_{32} 8$,… The next two terms are $\log_{64} 8$ and $\log_{128} 8$

General expression for the n^{th} term of this sequence is $\log_{2^n} 8 = 3/n$
View the diagram below to see how the work is done. I saw a pattern with 2^n and wrote down the relationship for the remaing sequence. The two boxes are interconnected as one side of work goes along with the other. Following, the logarithm rules of multiplication, division and powers and change of base.

$\log_{2^n} 8$
$= \log_8 8 - \log_8 2^n$
$= 1 - \log_8 2^n$
$= 1 – n\log_8 2$
$= 1 – n\,(1/3)$

$= \log_8 8 / \log_8 2^n$
$= 1 / n \log_8 2$
$= 1 / n\,(1/3)$
$= 1 / 1/3\, n$
$= (1/1) \times (3/n)$
$= 3/n$

- $\log_3 81$, $\log_9 81$, $\log_{27} 81$, $\log_{81} 81$,… The next two terms are $\log_{243} 81$ and $\log_{729} 81$

General expression for the n^{th} term of this sequence is $\log_{3^n} 81 = 4/n$. View the work below to understand.

$\log_{3^n} 81$
$= \log_{81} 81 - \log_{81} 3^n$
$= 1 - \log_{81} 3^n$
$= 1 – n \log_{81} 3$
$= 1 – n\,(1/4)$

$= \log_{81} 81 / \log_{81} 3^n$
$= 1 / n \log_{81} 3$
$= 1 / n\,(1/4)$
$= 1 / 1/4\, n$
$= (1/1) \times (4/n)$
$= 4/n$

- $\log_5 25$, $\log_{25} 25$, $\log_{125} 25$, $\log_{625} 25$,… The next two terms are $\log_{3125} 25$ and $\log_{15625} 25$

General expression for the n^{th} term of this sequence is $\log_{5^n} 25 = 2/n$. View the work below to understand.

$\log_{5^n} 25$ $= \log_{25} 25 - \log_{25} 5^n$ $= 1 - \log_{25} 5^n$ $= 1 – n \log_{25} 5$ $= 1 – n (1/2)$	$= \log_{25} 25 / \log_{25} 5^n$ $= 1 / n \log_{25} 5$ $= 1 / n (1/2)$ $= 1 / 1/2\ n$ $= (1/1)\ X\ (2/n)$ $= 2/n$

- $\log_m m^k$, $\log_{m^2} m^k$, $\log_{m^3} m^k$, $\log_{m^4} m^k$,...The next two terms are $\log_{m^5} m^k$ and $\log_{m^6} m^k$

General expression for the n^{th} term of this sequence is $\log_{m^n} m^k = k/n$. View the work below to understand.

$\log_{m^n} m^k$ $= \log_m m^k - \log_{m^k} m^n$ $= 1 - \log_{m^k} m^n$ $= 1 – n \log_{m^k} m$ $= 1 – n (1/k)$	$= \log_m m^k / \log_{m^k} m^n$ $= 1 / n \log_{m^k} m$ $= 1 / n (1/k)$ $= 1 / 1/k\ n$ $= (1/1)\ X\ (k/n)$ $= k/n$

- I found what the following terms were by following the patterns previous to the continuation of the sequence. By using GDC with the TI – 83 Plus I calculated the answers such as 5^6 for the third sequence.

Part 2 of Problem 1:

This involves the changing of the general expressions into the form p/q, where $p,q \in \mathbb{Z}$.
To do this you are solving each logarithm.
To review the mathematics involved view figure 1.

1.) The first sequence:
The general expression for the n^{th} term of this sequence is $\log_{2^n} 8 = 3/n$.
Therefore the expressions will be: 3/1, 3/2, 3/3, 3/4, 3/5,... and so forth.

2.) The second sequence:
The general expression for the n^{th} term of this sequence is $\log_{3^n} 81 = 4/n$.

Therefore, the expressions will be: 4/1, 4/2, 4/3, 4/4, ... and so forth.

3.) The third sequence:
The general expression for the n^{th} term of this sequence is $\log_{5^n} 25 = 2/n$.
Therefore, the expressions will be 2/1, 2/2, 2/3, 2/4, ...and so forth.

4.) The final sequence:
The general expression for the n^{th} term of this sequence is $\log_{m^n} m^k = k/n$.
Therefore, the expressions will be k/1, k/2, k/3… and so forth.

- To justify my answers with technology I used GDC with the TI – 83 Plus.
- I converted certain decimals to fractions using the calculator.

Problem 2: Now calculate the following giving your answers in the form p/q where $p,q \in \mathbb{Z}$.

- $\log_4 64 = 3/1$ $\quad \log_8 64 = 2/1$ $\quad \log_{32} 64 = 6/5$

- $\log_7 49 = 2/1$ $\quad \log_{49} 49 = 1/1$ $\quad \log_{343} 49 = 2/3$
- $\log_{1/5} 125 = {}^-3/1$ $\quad \log_{1/125} 125 = {}^-1/1$ $\quad \log_{1/625} 125 = {}^-3/4$
- $\log_8 512 = 3/1$ $\quad \log_2 512 = 9/1$ $\quad \log_{16} 512 = 9/4$

Part 2 of Problem 2:
Describe how to obtain the third answer in each row from the first two answers.
If $\log_a x = c$ and $\log_b x = d$
therefore, the general statement that expresses $\log_{ab} x$ in terms of c and d is:

$\log_{ab} x = (cd)/(c + d)$ I arrived to this statement by combing all of the laws together.

Restrictions: $a > 0$, $b > 0$, $ab \neq 1$, $c \neq -d$

$ab \neq 1$ because otherwise

Example of validity:
$\log_3 81 = 4$, $\log_{1/3} 81 = -4$, $\log_1 81 = ?$
$(cd)/(c + d)$ would be $(4\mathbf{X}{-4}) / (4 + {}^-4) = -16/0$, which is undefined.
1 raised to any power is always equal to 1, and will never equal 81.
This also shows why $c \neq -d$, because you can't divide by 0.

Part 3 of Problem 2:
Create two or more examples that fit the pattern above.

- $\text{Log}_3 81$ $\quad \log_9 81$ $\quad \log_{27} 81$
  $\log_3 81 = 4$, $\quad \log_9 81 = 2$ $\quad \log_{27} 81 = 4/3$

  Therefore $(cd) / (c + d) = (4\mathbf{X}2) / (4 + 2) = 8/6 = 4/3$

- $\log_4 64$ $\quad \log_{64} 64$ $\quad \log_{256} 64$
  $\log_4 64 = 3$ $\quad \log_{64} 64 = 1$ $\quad \log_{256} 64 = ¾$

  Therefore $(cd) / (c + d) = (3\mathbf{X}1) / (3 + 12) = 3/4 = ¾$

Generalizations:

- You can find the logarithms of positive and negative integers.
- Logarithms follow exponential rules being exponents themselves.

Limitations:

- Restrictions for certain variables were explained within the problem itself, therefore, view corresponding figures to see those circumstances.
- You can find the log of fractions and square roots as shown in figure 8.
- If trying to find the logarithm of a negative number the result will hold an imaginary solution rather than a "real" number.