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Lacsap's Fractions : Internal Assessment

IB Math SL Type 1

Aim: In this task you will consider a set of numbers that are presented in a symmetrical pattern.

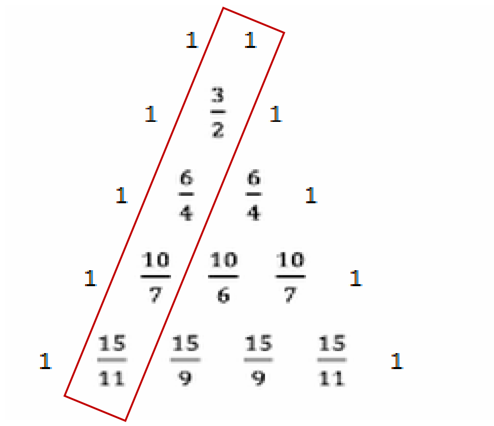


Figure 1 : Lacsap's Fractions

The first five rows of numbers are shown above. In order to find the numerator of the sixth row, I will use the numbers that go down the triangle diagonal, as shown from the highlighted fractions above. Hence the numerators are:

1 3 6 10 15

Row	Numerator	1^{st} difference	2^{nd} difference
1	1	2	1
2	3	3	
3	6	4	1
4	10		
5	15	5	1

Figure 2 : Table showing relationship between n rows and numerator

The table above shows the relationship between row and numerator. The first difference between the numerator in row 1 and 2 was **2**, between row 2 and 3 was **3**, and so forth (2, 3, 4, 5). The second difference for each row number is 1, hence the equation for the numerator is a geometric sequence. Therefore, the find the equation of the sequence, the quadratic formula, $\mathbf{y = ax^2 + bx + c}$ should be used, where $\boldsymbol{y}$ is the numerator and $\boldsymbol{x}$ is the row number.

To find this general statement for the numerator, I will calculate the values of a and b using simultaneous equations (substitution method):

Using the values from the table: **x = 2** and **y = 3** (second row)

Substitute into the quadratic formula (**c** is disregarded), and make **b** the subject:

$3 = a\,(2)^2 + b(2) + 0$

$3 = 4a + 2b$

$b = -2a + 1.5$

Using the values from the third row : **x = 3** and **y = 6**

$6 = a\,(3)^2 + b(3) + 0$

$6 = 9a + 3b$

Substitute **b = -2a + 1.5**,

$6 = 9a + 3(-2a + 1.5)$

$6 = 9a - 6a + 4.5$

$3a = 1.5$

$a = 0.5$

Therefore,

$b = -2(0.5) + 1.5$

$b = -1 + 1.5 = 0.5$

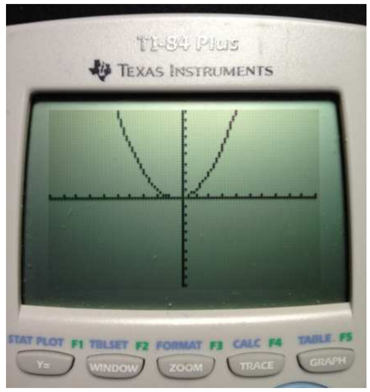
We have come to a general statement for the nth term of the numerator:

$$S_n = 0.5n^2 + 0.5n$$

Re-written:

$$S_n = \frac{n^2 + n}{2}$$

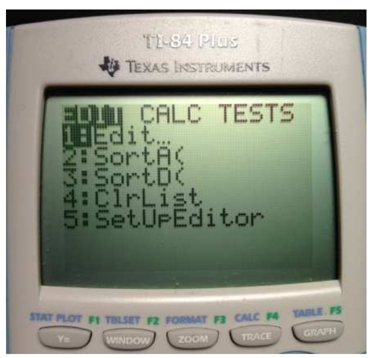
Another method involving the use of the TI -83 Plus Calculator can be used. By using the formula of ***0.5n²* + *0.5n*** achieved from the method above, I entered it into the GDC.



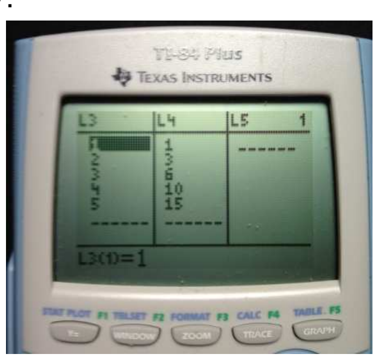
When the formula was entered into the GDC, the graph above was shown. This proves that it is a quadratic equation.

Therefore, the equation can be solved by using the Quadratic Regression '**QuadReg**' function. The steps of this have been shown through a series of camera shots of the calculator.

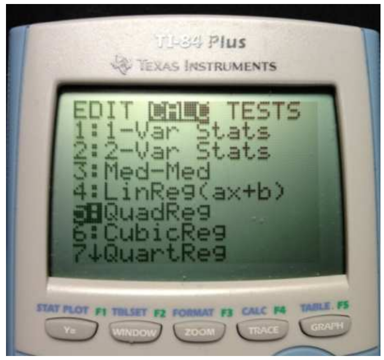
1. Press the **STAT** button on the calculator, go to **EDIT**.



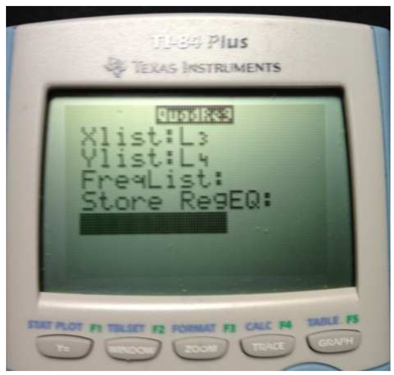
Once in **EDIT**, two empty columns will appear. Input the row number, ***n***, in the first column, and the numerator in respect with the row number in the second column, as shown below:



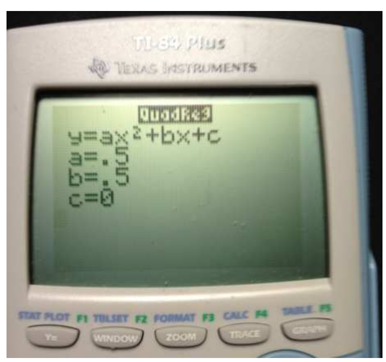
Once this is done, exit the screen, and press **STAT** again and scroll right to **CALC**.



Scroll down to **QuadReg**, and hit enter:



The screen above will appear. Choose the two lists with the row number and the numerator (in this case, L_3 and L_4), and press Calculate.



The calculator will display the values above. From t he manual calculations, the calculator also shows the formula as being $\boldsymbol{0.5n^2 + 0.5n}$, hence the formula is correct.

To validate the general statement for S_n, using n = 4 and n = 5 as examples:

$S_n = \frac{4^2+4}{2}$ = 10 Correct, matches with *Figure 1*

Sn = $\frac{5^2+5}{2}$ = 15 Correct, matches with *Figure 1*

Hence, the numerator for the sixth row is:

$S_n = \frac{6^2+6}{2}$ = **21**

I will plot a graph with the relation between the row number, ***n***, and the numerator in each row.

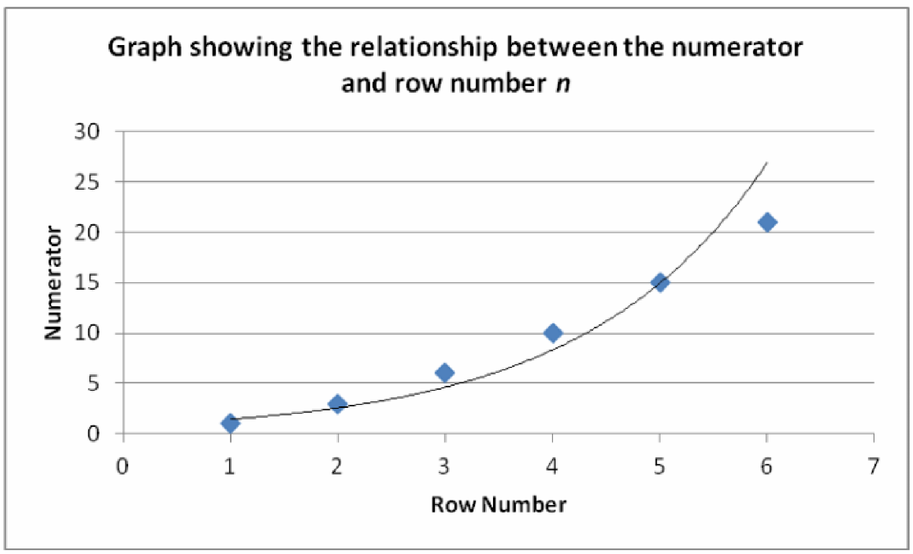


Figure 3 : Graph showing correlation between numerator and row number

Note: The graph above was created using Microsoft Excel

The above shows the correlation from row 1 to row 6. As the trend line is no linear, it suggests that the equation is exponential, due to the fact that the equation is a quadratic (***0.5n²* + *0.5n***). Therefore, the general statement is correct.

In order to find the sixth and seventh rows, the denominator of each element must be found in addition to the numerator in each row. To find the denominator in each element of each r ow, the relationship between the numerator and the denominator must be observed. The "1"s shown in Figure 1 are discarded as they are not needed and do not affect the relationship. This will limit the number of elements in a certain row to being *n - 1* (i.e. row 2 has only 1 element). To find the denominator for each element, the denominator of the first element of each row was observed first as shown in the diagram on the next page.

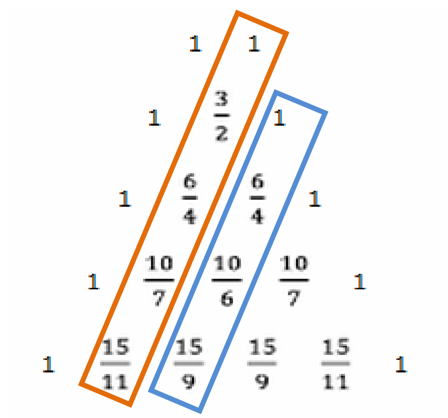


Figure 4 : Relationship between numerator and denominator of element 1 (orange) and element 2 (blue)

The difference between the numerator and denominator was observed for element 1 (highlighted in orange above) . The difference was recorded as 1, 2, 3 and 4 for row 2, 3, 4, 5 respectively. Using the numerator as the base for finding the denominator, the initial equation is formed:

$$Numerator - (x) = Denominator \qquad \text{Equation 1}$$

x is the difference between the numerator and denominator, formed by using the row and element numbers. As the difference of the denominator of element 1 increases by 1 for every consecutive row, a statement can be made that the numerator minus 1 multiplied by the row number minus 1 gives the denominator for element 1 (***n*** as the row number):

$$Numerator - 1(n - 1) = Denominator \qquad \text{Equation 2}$$

Now looking at the second row (highlighted in blue), the differences between the numerator and the denominator of the second element was recorded as 2, 4, and 6. For every consecutive row, the difference between the numerator and the denominator increases by 2. A general statement fo r this is shown below:

$$Numerator - 2(n - 2) = Denominator \qquad \text{Equation 3}$$

Now that the two equations have been formulated for elements 1 and 2, the similarities for both elements was observed. In Equation 2, the "1" in the x part of the equation is equivalent to the el ement number and as for the equation 3, "2" in the ***x*** part of the equation is also equivalent to the element number. Hence, the ***x*** part of the equation can now be subs tituted by the element number, ***r***, to form the general statement shown below:

$$Numerator - r(n - r) = Denominator \qquad \text{Equation 4}$$

or

$$\frac{n^2+n}{2} - r(n - r) = Denominator$$

To find the sixth and seventh row, the general statement of the numerator and the denominator must be combined to create the general statement for the element,
$E_n(r)$, as shown below:

$$E_n(r) = \frac{\frac{n^2+n}{2}}{\frac{n^2+n}{2} - r(n-r)} \qquad \text{Equation 5}$$

Using the general statement above, each element can be found by substituting ***n*** for the row number, and ***r*** for element number, starting from ***r = 1*** up to ***r = n - 1*** according to the limitations. Using the equation as the sixth row:

$$E_6(1) = \frac{\frac{6^2+6}{2}}{\frac{6^2+6}{2} - 1(6-1)} \quad \ldots \quad E_6(5) = \frac{\frac{6^2+6}{2}}{\frac{6^2+6}{2} - 5(6-5)}$$

Using the calculations above, the sixth row comes out as shown below

$$\frac{21}{16} \quad \frac{21}{13} \quad \frac{21}{12} \quad \frac{21}{13} \quad \frac{21}{16}$$

Knowing that "1" was discarded while doing the calculations, the "1" must be added back into the row at the beginning and the end. The entire row is shown below:

$$1 \quad \frac{21}{16} \quad \frac{21}{13} \quad \frac{21}{12} \quad \frac{21}{13} \quad \frac{21}{16} \quad 1$$ Row 6

The seventh row is also found by doing the same as above:

$$E_7(1) = \frac{\frac{7^2+7}{2}}{\frac{7^2+7}{2} - 1(7-1)} \quad \ldots \quad E_7(6) = \frac{\frac{7^2+7}{2}}{\frac{7^2+7}{2} - 6(7-6)}$$

The seventh row comes out as shown below:

$$\frac{28}{21} \quad \frac{28}{18} \quad \frac{28}{16} \quad \frac{28}{16} \quad \frac{28}{18} \quad \frac{28}{21}$$

Again, putting the "1" back into the beginning and en d of the row:

$$1 \quad \frac{28}{21} \quad \frac{28}{18} \quad \frac{28}{16} \quad \frac{28}{16} \quad \frac{28}{18} \quad \frac{28}{21} \quad 1$$ Row 7

In order to test the **validity** of the general statement, I will use the statement to find additional rows.

By using the same method to find the sixth and seventh rows, the eighth, ninth, tenth rows were found to validate the validity of the general statement using equation 5.

$$E_8(1) = \frac{\frac{8^2+8}{2}}{\frac{8^2+8}{2} - 1(8-1)} \quad \ldots \quad E_8(7) = \frac{\frac{8^2+8}{2}}{\frac{8^2+8}{2} - 7(8-7)}$$

$$E_9(1) = \frac{\frac{9^2+9}{2}}{\frac{9^2+9}{2} - 1(9-1)} \quad \ldots \quad E_9(8) = \frac{\frac{9^2+9}{2}}{\frac{9^2+9}{2} - 8(9-8)}$$

$$E_{10}(1) = \frac{\frac{10^2+10}{2}}{\frac{10^2+10}{2} - 1\,(10-1)} \ldots \quad E_{10}(9) = \frac{\frac{10^2+10}{2}}{\frac{10^2+10}{2} - 9\,(10-9)}$$

Using the **eighth, ninth, tenth rows**, the rows come out as shown below:

$$1 \quad \frac{36}{29} \quad \frac{36}{24} \quad \frac{36}{21} \quad \frac{36}{20} \quad \frac{36}{21} \quad \frac{36}{24} \quad \frac{36}{29} \quad 1$$

$$1 \quad \frac{45}{37} \quad \frac{45}{31} \quad \frac{45}{27} \quad \frac{45}{25} \quad \frac{45}{25} \quad \frac{45}{27} \quad \frac{45}{31} \quad \frac{45}{37} \quad 1$$

$$1 \quad \frac{55}{46} \quad \frac{55}{39} \quad \frac{55}{34} \quad \frac{55}{31} \quad \frac{55}{30} \quad \frac{55}{31} \quad \frac{55}{34} \quad \frac{55}{39} \quad \frac{55}{46} \quad 1$$

Again, replacing the "1"s at the beginning and end of each row, as shown below:

$$1 \quad \frac{36}{29} \quad \frac{36}{24} \quad \frac{36}{21} \quad \frac{36}{20} \quad \frac{36}{21} \quad \frac{36}{24} \quad \frac{36}{29} \quad 1 \qquad \text{Row 8}$$

$$1 \quad \frac{45}{37} \quad \frac{45}{31} \quad \frac{45}{27} \quad \frac{45}{25} \quad \frac{45}{25} \quad \frac{45}{27} \quad \frac{45}{31} \quad \frac{45}{37} \quad 1 \qquad \text{Row 9}$$

$$1 \quad \frac{55}{46} \quad \frac{55}{39} \quad \frac{55}{34} \quad \frac{55}{31} \quad \frac{55}{30} \quad \frac{55}{31} \quad \frac{55}{34} \quad \frac{55}{39} \quad \frac{55}{46} \quad 1 \qquad \text{Row 10}$$

Hence, the general statement is valid and works with additional rows **above 7**.

During the calculation of the numerator and the denominator of each element, the "1" at the beginning and the end of each row was discarded before making the calculations. This becomes a limitation to the application for using the general equation that was derived from a specific element in a specific row. Since the first row only consisted of "1", and hence the "1" was discarded from calculations, the general equation cannot be used to calculate the elements in the first row. Also due to the fact that "1" is discarded, each element in each row is also decreased by 2, therefore the element number does not start from 0 (the beginning being "1"), but starts at 1. This means, to calculate a specific element in a specific row, ***r*** must be **greater than or equal** to 2, and ***n*** must be **greater than or equal** to 1.

I have arrived to the general statement through a step by step process, firstly finding the general statement for the numerator in each row. This was validated through the use of technology (GDC; **QuadReg**), and tested before moving onto the next step of finding the general statement for the denominator. By observing the relationship between the numerator and denominator, I found out that there was a relationship between the differences. By splitting the elements and using the dif ferences as a base, I noticed that the general statement for the denominator had a relationship with the element number and row numbers, hence a statement was conjured from these areas. By inputting the general statement of the numerator, I came up with th e general statement for $E_n (r)$. This statement was proven correct after validating the statement with the findings of additional rows (greater than 7).