

# **THE KOC** **SNOWFLAKE**

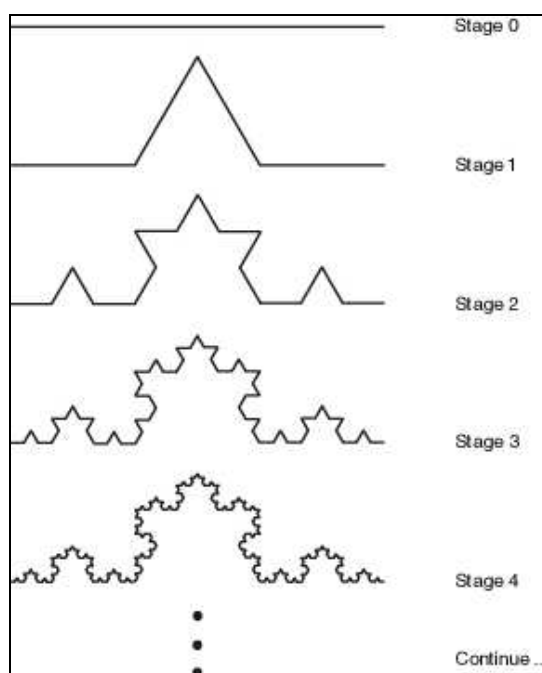
## **A MATHEMATICS PORTFOLIO**

**MADE BY:-  
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IB-HL**

## INTRODUCTION

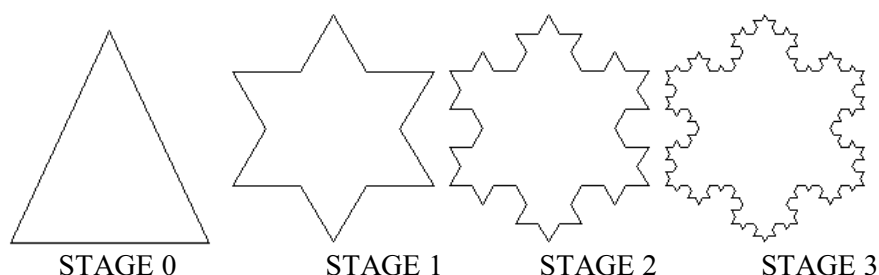
The Koch Snowflake was created by the Swedish mathematician Niels Fabian Helge von Koch. In his 1904 paper entitled *"Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire"* he used the Koch Snowflake to show that it is possible to have figures that are continuous everywhere but differentiable nowhere.

In order to reach the Koch snowflake, we need to start with a Koch curve. A Koch curve is a straight line which is divided into three parts and an equilateral triangle is constructed keeping the base as the middle part of the divided line segment.



Finally, the base is removed which gives us the first iteration of the Koch curve (as shown in the figure above).

From the Koch curve, comes the Koch snowflake. Now, instead of one line it starts with an equilateral triangle. The steps in Koch curve are applied now to each side of the equilateral triangle, which ultimately gives the shape of a "snowflake".



## **PROCEDURE & OBSERVATION**

### ASSUMPTIONS –

Let  $N_n$  = Number of sides,  $L_n$  = the length of a single side,  $P_n$  = the length of the perimeter,  $A_n$  = the area of the snowflake; at the  $n$ th stage.

In this portfolio, the sign ‘\*’ indicates multiplication whereas the sign ‘/’ indicates division.

**Q1. Using an initial side length of 1, create a table that shows the values of  $N_n$ ,  $L_n$ ,  $P_n$ , and  $A_n$  for  $n = 0, 1, 2$  and  $3$ . Use exact values in your results. Explain the relationship between successive terms in the table for each quantity  $N_n$ ,  $L_n$ ,  $P_n$ , and  $A_n$ .**

Iterations (n)	Number of sides ( $N_n$ )	Length of one side ( $L_n$ )	Perimeter ( $P_n$ )	Area ( $A_n$ )
0	3	1	3	0.433013
1	12	0.333333	4	0.577350
2	48	0.111111	5.333333	0.641500
3	192	0.037037	7.111111	0.670011

Relationship between successive terms for each quantity:-

$N_n \rightarrow$  Every successive quantity increases by a factor of 4, i.e. it is a geometric sequence and the ratio between successive terms remains the same.

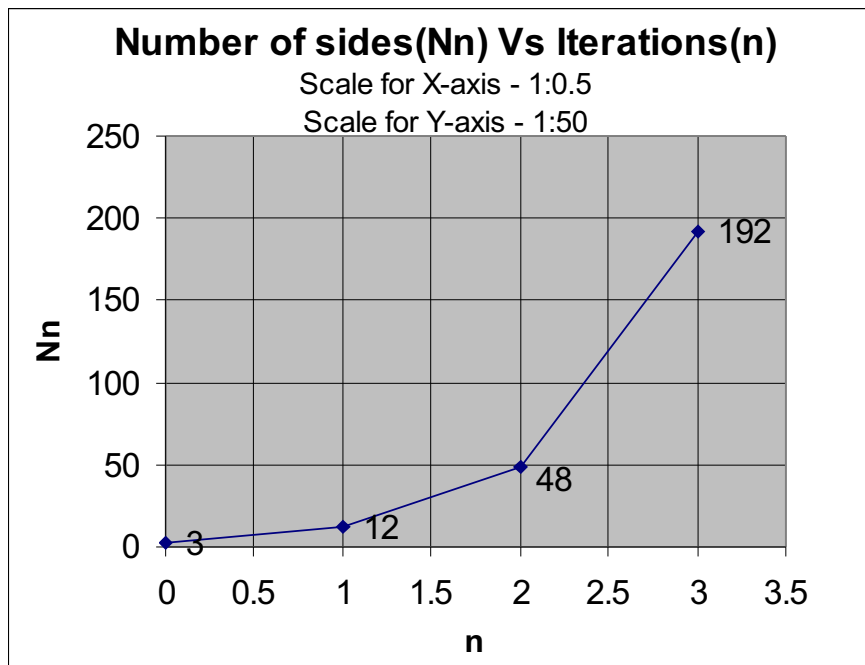
$L_n \rightarrow$  Every successive quantity is  $1/3^{\text{rd}}$  of the previous length thus this is also a geometric sequence, i.e. the ratio between two successive terms remains the same.

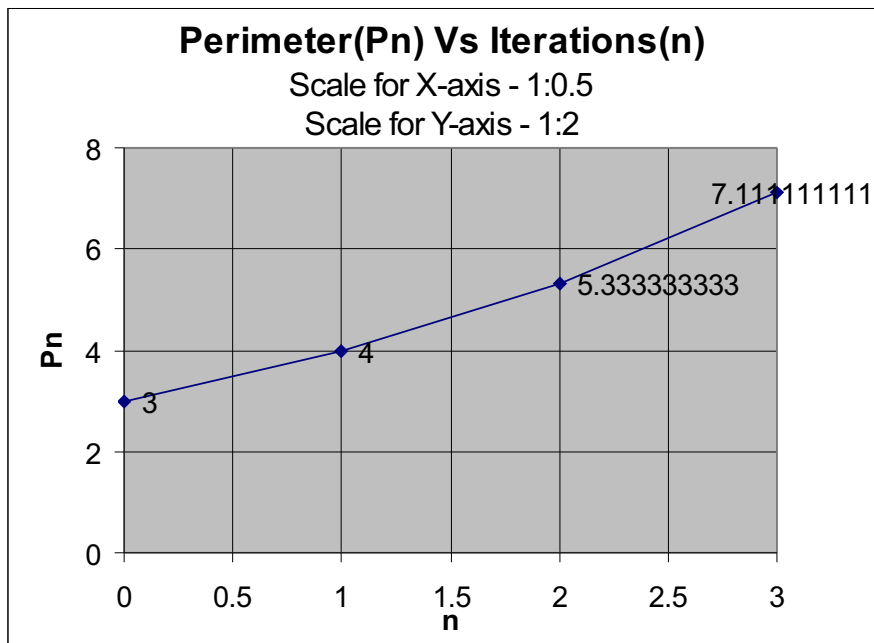
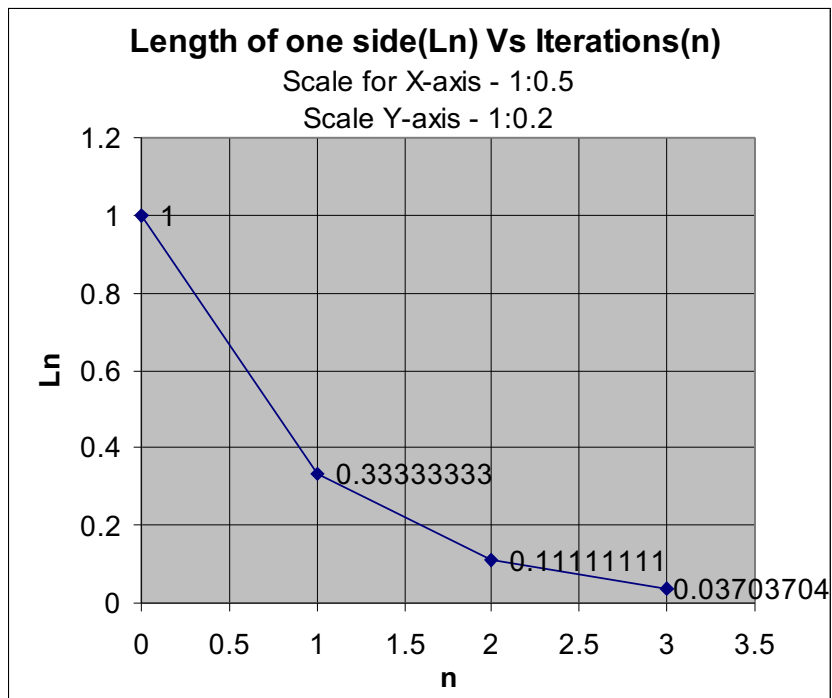
$P_n \rightarrow$  This also follows a geometric pattern as every successive quantity increases by a factor of  $4/3$  and the ratio here as well remains the same.

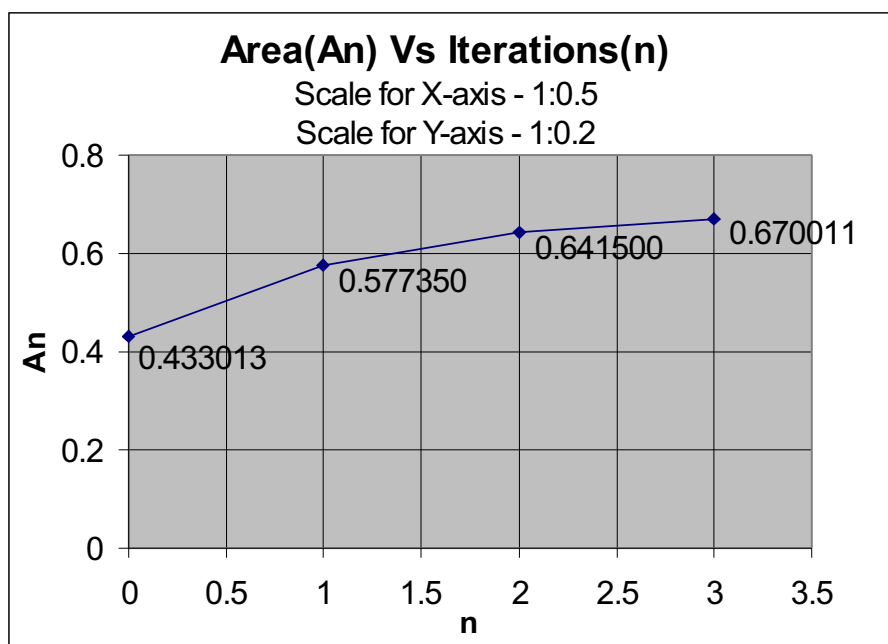
$A_n \rightarrow$  The relationship between successive terms of area is very complex and thus it is explained in an appropriate manner in Q3.

**Q2. Using a GDC or a suitable graphing software package, create graphs of the four sets of values plotted against the value of  $n$ . Provide separate printed output for each graph.**

Using Microsoft Excel, I have created graphs of the four sets of values based on the values in the table created in Q1:-







Thus, the graphs that I have made above have a general formula (explained in Q3 below) which will satisfy the graph at any value of  $n$ .

**Q3. For each of the graphs above, develop a statement in terms of  $n$  that generalizes the behavior shown in its graph. Explain how you arrived at your generalizations. Verify that your generalizations apply consistently to the sets of values produced in the table.**

### Number of sides of the Koch Snowflake

Iteration ( $n$ )	Number of sides ( $N_n$ )
0	3
1	12
2	48
3	192

From the table above we can see that there is an increase in the number of sides at every stage by a factor of 4. At every succeeding stage, the number of sides is 4 times the number of sides in the preceding stage. This means that the number of sides of the snowflake follow a geometric sequence. With the help of this concept of geometric progression, we can formulate a general formula for the number of

the sides of the snowflake. The general formula for a geometric progression is as follows –

$$U_n = U_0 * R^n$$

Where R = ratio between two successive terms,  $U_0 = 1^{\text{st}}$  term, and  $U_n = \text{nth}$  term of the sequence. Substituting  $U_n$  with  $N_n$  can give us the general formula for the number of sides of the snowflake which is –

$$N_n = N_0 * R^n$$

$$= 3 * 4^n$$

Since, R = 4 and  $N_0 = 3$  (as found out from the table above).

Verification –

For n = 0

$$N_0 = 3 * 4^0 = 3 * 1 = 3$$

For n = 1

$$N_1 = 3 * 4^1 = 3 * 4 = 12$$

For n = 2

$$N_2 = 3 * 4^2 = 3 * 16 = 48$$

For n = 3

$$N_3 = 3 * 4^3 = 3 * 64 = 192$$

Thus, the generalizations for  $N_n$  made above apply consistently to the sets of values given in the table and also satisfy the graph made above.

### **Length of one side of the Koch Snowflake**

Iteration (n)	Length of one side ( $L_n$ )
0	1
1	0.333333
2	0.111111
3	0.037037

From the table above we can observe that at every succeeding stage, the length of one side is  $1/3^{\text{rd}}$  the length of one side of the preceding stage. This means that there

is again a geometric sequence that can be seen. Keeping the formula for geometric progression ( $U_n = U_0 * R^n$ ) in mind, a general formula can be formed for the length of one side of the koch snowflake –

$$\begin{aligned} L_n &= L_0 * R^n \\ &= 1 * (1/3)^n \\ &= (1/3)^n \end{aligned}$$

Since  $L_0 = 1$ , and  $R = (1/3)$  (as seen from the table above)

Verification –

For  $n = 0$

$$L_0 = (1/3)^0 = 1/1 = 1$$

For  $n = 1$

$$L_1 = (1/3)^1 = 1/3 = 0.333333$$

For  $n = 2$

$$L_2 = (1/3)^2 = 1/9 = 0.111111$$

For  $n = 3$

$$L_3 = (1/3)^3 = 1/27 = 0.037037$$

Thus, the generalizations for  $L_n$  made above apply consistently to sets of values produced in the table and it also satisfies the graph above.

### Perimeter of the Koch Snowflake

Iteration (n)	Perimeter ( $P_n$ )
0	3
1	4
2	5.333333
3	7.111111

To find the perimeter of any polygon, the easiest and the most convenient method is to multiply the number of the sides of the given polygon with the length of one side as the snowflake is a sort of a regular polygon. The same is the case of the koch snowflake. If we multiply the number of sides ( $N_n$ ) with the length of one side ( $L_n$ ), we would obtain the perimeter of the snowflake at every stage. With the



help of this concept, we arrive at deriving a general formula to find the perimeter which is –

$$\begin{aligned} & N_n * L_n \\ &= 3 * 4^n * (1/3)^n \\ &= 3 * (4/3)^n \end{aligned}$$

Now here, we can again observe that the perimeter can also be found out using the method of geometric progression. The perimeter of the koch snowflake, like its number of sides ( $N_n$ ) and the length of one side ( $L_n$ ), follows a geometric pattern which can be written in the following way –

$$\begin{aligned} P_n &= P_0 * R^n \\ &= 3 * (4/3)^n \end{aligned}$$

Since,  $P_0 = 3$  and  $R = 4/3$  (as can be seen from the table above)

Verification –

For  $n = 0$

$$P_0 = 3 * (4/3)^0 = 3 * 1 = 3$$

For  $n = 1$

$$P_1 = 3 * (4/3)^1 = 4 * 1 = 4$$

For  $n = 2$

$$P_2 = 3 * (4/3)^2 = 16/3 = 5.333333$$

For  $n = 3$

$$P_3 = 3 * (4/3)^3 = 64/9 = 7.111111$$

Thus, the generalizations for  $P_n$  made above apply consistently to sets of values produced in the table and also verify the graph.

### Area of the Koch Snowflake

Iterations (n)	Area ( $A_n$ )
0	0.433013
1	0.57735
2	0.6415
3	0.670011

The area of any equilateral triangle is given by the formula  $(\sqrt{3}/4)*s^2$ , where  $s$  is the length of a side of the equilateral triangle. As we can see from the table, the initial areas of the snowflake at stage 0, i.e.,  $A_0 = \sqrt{3}/4 = 0.43301$ . At every succeeding stage, a certain number of additional triangles add to the previous figure of the snowflake, increasing the area. For example,

At stage 1,

$$\text{Area of the snowflake} = \text{Area of the snowflake in previous stage 0} + \text{Area of additional triangles added in stage 1}$$

Thus, the area of additional triangles at each stage can found out by multiplying the area of one additional triangle with the total number of additional triangles. At every succeeding stage, a triangle is added on every side of the snowflake (as explained in the introduction of my portfolio). Thus, the number of additional triangles added at every stage is equal to the number of sides ( $N_n$ ) of the snowflake in the preceding stage. Hence, we can find a general formula for the number of additional triangles, which is -

$$3*4^{n-1}$$

This is because the number of sides is given by the formula  $3*4^n$ .

Now the area of each additional triangle will be  $(\sqrt{3}/4)*s^2$ , where  $s$  is the length of one side of the equilateral additional triangle. The length of one side at  $n$ th stage, as shown above is  $(1/3)^n$ . Thus, the area of each additional triangle will be -

$$\begin{aligned} &= (\sqrt{3}/4)*[(1/3)^n]^2 \\ &= (\sqrt{3}/4)*(1/3^{2n}) \\ &= (\sqrt{3}/4)/3^{2n} \end{aligned}$$

Hence, the area of additional triangles at each stage is given by -

$$\begin{aligned} &= (\sqrt{3}/4)*3*4^{n-1}/3^{2n} \\ &= (\sqrt{3}*3*3^{2n})*(4^{-1}*4^{n-1}) \\ &= 3^{(3/2)-2n}*4^{n-2} \end{aligned}$$

Now, if the area of the snowflake in the present stage is  $A_n$ , then the area of the snowflake in the preceding can be written as  $A_{n-1}$ . Thus, now I can finally arrive at the general formula for the area of the koch snowflake, which is -

$$A_n = A_{n-1} + (3^{(3/2)-2n}*4^{n-2})$$

Verification -

For  $n = 1$

$$A_1 = A_0 + (3^{(3/2)-2*1}*4^{(1-2)})$$

$$\begin{aligned}
 &= (\sqrt{3}/4) + (3^{(-1/2)} * 4^{-1}) \\
 &= 0.433013 + 1/(\sqrt{3} * 4) \\
 &= 0.433013 + 0.144347 \\
 &= 0.577350
 \end{aligned}$$

For  $n = 2$

$$\begin{aligned}
 A_2 &= A_1 + (3^{(3/2) - 2 * 2} * 4^{(2 - 2)}) \\
 &= 0.57735 + (3^{(-5/2)} * 1) \\
 &= 0.57735 + 1/(3^{(5/2)}) \\
 &= 0.57735 + 0.06415 \\
 &= 0.641500
 \end{aligned}$$

For  $n = 3$

$$\begin{aligned}
 A_3 &= A_2 + (3^{(3/2) - 2 * 3} * 4^{(3 - 2)}) \\
 &= 0.6415 + (3^{(-9/2)} * 4) \\
 &= 0.6415 + 4/(3^{(9/2)}) \\
 &= 0.6415 + 0.028511 \\
 &= 0.670011
 \end{aligned}$$

Thus, the generalizations for  $A_n$  made above apply consistently to sets of values produced in the table except for  $A_0$  because if I put  $n = 0$ , then the formula of area will not satisfy the graph. Hence, the generalizations that I have made is not true for  $n=0$ .

**Q4. Investigate what happens at  $n = 4$ . Use your conjectures from step 3 to obtain values for  $N_4$ ,  $L_4$ ,  $P_4$ , and  $A_4$ . Now draw a large diagram of one “side” (that is, one side of the original triangle that has been transformed) of the fractal at stage 4 and clearly verify your predictions.**

Now, after I have made the generalizations, I will try to use those general formulas made and verified in Q3. to investigate what happens at when  $n = 4$ .

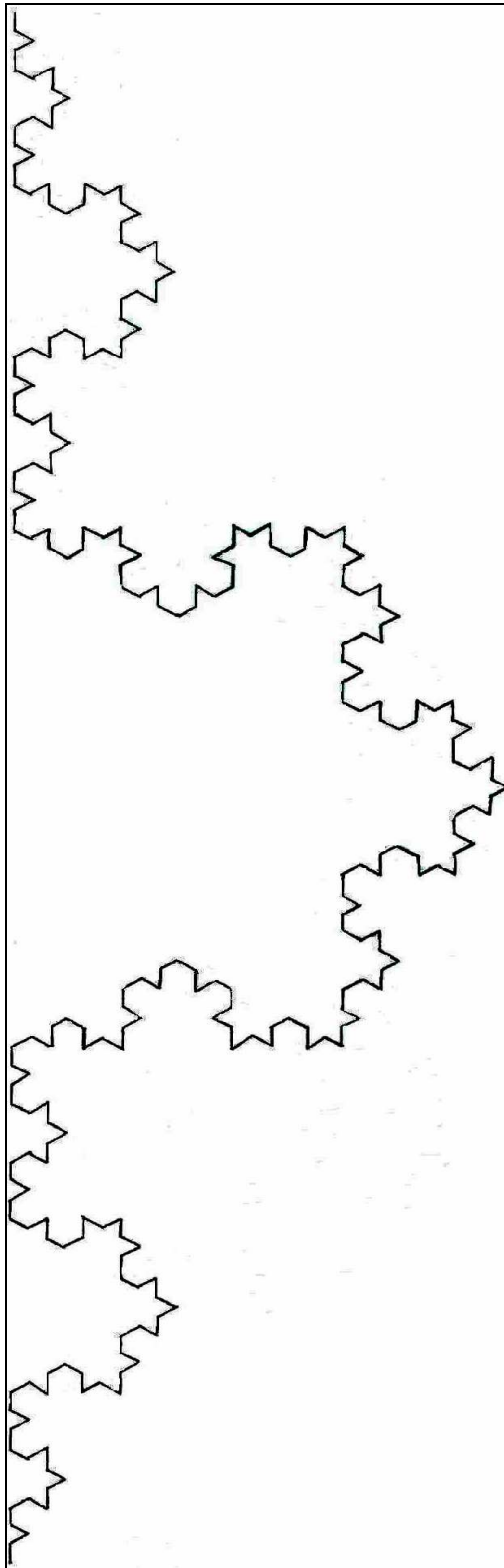
$$\begin{aligned}
 N_4 &= 3 * 4^4 \\
 &= 3 * 256 \\
 &= 768
 \end{aligned}$$

$$\begin{aligned}
 L_4 &= (1/3)^4 \\
 &= 1/81 \\
 &= 0.012346
 \end{aligned}$$

$$\begin{aligned} P_4 &= 3 \cdot (4/3)^4 \\ &= 256/27 \\ &= 9.481481 \end{aligned}$$

$$\begin{aligned} A_4 &= A_3 + (3^{(3/2) - 2 \cdot 4} \cdot 4^{(4-2)}) \\ &= 0.670011 + (3^{(-13/2)} \cdot 4^2) \\ &= 0.670011 + 16/3^{(13/2)} \\ &= 0.670011 + 0.012672 \\ &= 0.682683 \end{aligned}$$

Thus, the generalizations I had made in question 3 have been proved to satisfy the values at stage 4. Below is a large diagram of one “side” of the fractal at stage 4.



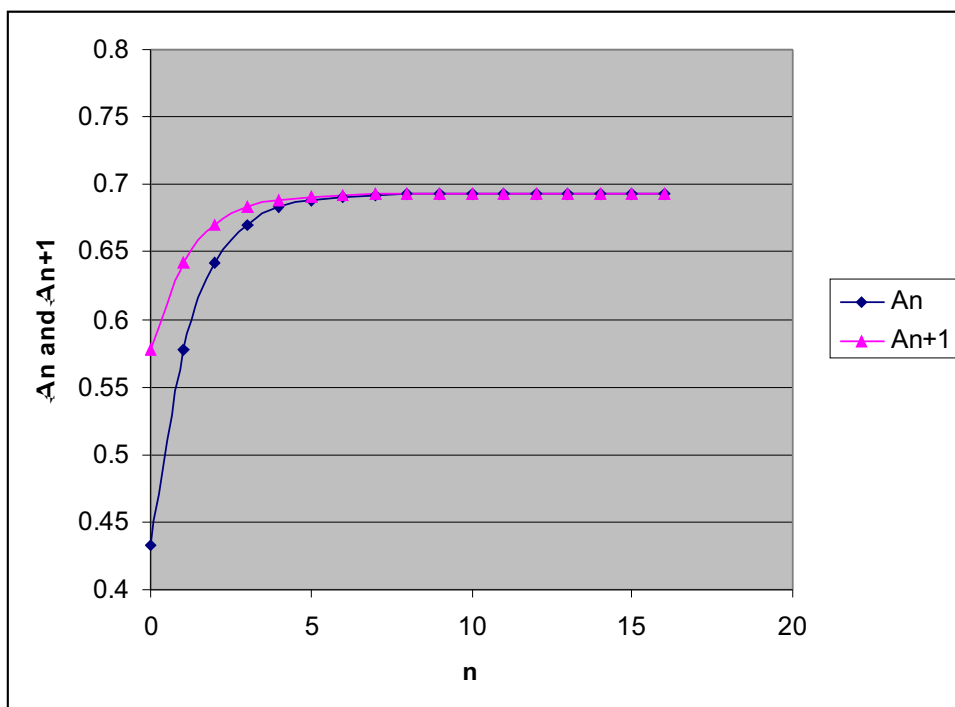
## **STAGE 4 of the Koch Snowlake**

**Q5. Using the relationship between successive terms of  $N_n$ ,  $L_n$ ,  $P_n$ , and  $A_n$ , use a spreadsheet or similar software to find the values of each up to a value for  $n$  where  $A_{n+1}$  equals  $A_n$  to six places of decimals. What is the value of  $n$  at this point?**

Using Microsoft Excel as my spreadsheet, I have worked out the value for  $n$  where  $A_{n+1} = A_n$  to six places of decimals (The values were worked out till the seven places of decimals and were rounded off to six places of decimals).

n	$A_n$	$A_{n+1}$
0	0.433013	0.577350
1	0.577350	0.641500
2	0.641500	0.670011
3	0.670011	0.682683
4	0.682683	0.688315
5	0.688315	0.690818
6	0.690818	0.691930
7	0.691930	0.692425
8	0.692425	0.692644
9	0.692644	0.692742
10	0.692742	0.692786
11	0.692786	0.692805
12	0.692805	0.692813
13	0.692813	0.692817
14	0.692817	0.692819
15	0.692819	0.692820
16	0.692820	0.692820

The value of  $n$  at which  $A_{n+1} = A_n$  is found out to be **16**. Thus, at this value  $A_{16} = A_{16+1}$ , or  $A_{16} = A_{17}$  (to six places of decimals). Below is given a graph to show how the difference in values of  $A_n$  and  $A_{n+1}$  decreases and they come closer and become equal at  $n=16$ .



**Q6. Explain what happens to the perimeter and area as  $n$  gets very large. What conclusion can you make about the area as  $n \rightarrow \infty$ ? Comment on your results.**

As  $n$  increases, we should observe that the values of the length of the perimeter and the area of the Koch snowflake increases as well but the extent to which it increases is not defined.

In the case of the length of the perimeter of the snowflake, the value of successive terms increases by a ratio of  $4/3$ . Thus, as  $n$  tends to infinity the perimeter of the snowflake will also reach infinity in due course.

In case of the area of the snowflake, the general formula of area is given by-

$$A_n = A_{n-1} + (\sqrt{3}/4) * 3 * 4^{n-1} / 3^{2n}$$

$$= A_{n-1} + A_0 * 3 * 4^{n-1} / 9^n$$

Multiplying  $A_0 * 3 * 4^{n-1} / 9^n$  by  $3 * (1/3)$ , we get

$$= A_{n-1} + A_0 * (4/9)^{n-1} / 3$$

Now in case of  $n=1$

$$A_1 = A_0 + A_0 * (4/9)^{1-1} / 3$$

$$= A_0 [1 + \{(1/3) * (4/9)^0\}]$$

When  $n=2$

$$\begin{aligned} A_2 &= A_0[1 + \{(1/3)*(4/9)^0 + (1/3)*(4/9)^1\}] \\ &= A_0[1 + (1/3)*\{1 + (4/9)^0 + (4/9)^1\}] \end{aligned}$$

Thus,

$$A_n = A_0[1 + (1/3)*\{1 + (4/9)^0 + (4/9)^1 + \dots + (4/9)^{n-1}\}]$$

Since, the values inside the curly brackets form a geometric progression thus as  $n \rightarrow \infty$ , we can use the formula –

$$S_\infty = U_1/(1-R)$$

Where  $U_1$  = first term, and  $R$  = ratio between two successive terms.

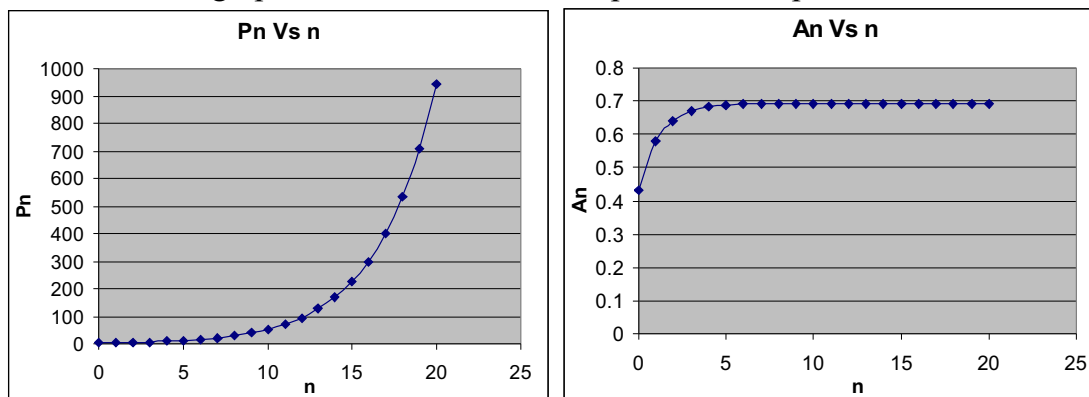
Hence,

$$\begin{aligned} \text{as } n \rightarrow \infty; A_n &= A_0[1 + (1/3)*\{1/(1-(4/9))\}] \\ &= (8/5)*A_0 \\ &= 1.6*A_0 \end{aligned}$$

Hence, we come to a great conclusion that the area of the Koch snowflake is actually 60% more than that of the area of the original triangle.

Finally, I come to a great conclusion that as  $n \rightarrow \infty$ , the Koch snowflake has infinite perimeter but finite area.

Below are two graphs to visualize the concept of infinite perimeter and finite area.



**Q7. Using your general expression for  $A_n$  and the iterative relationship between  $A_{n+1}$  and  $A_n$  already established in step 1 above, prove the general expression for  $A_n$  by induction.**

The general formula for the area of the Koch snowflake is –

$$A_n = A_{n-1} + (3^{(3/2)-2n} * 4^{n-2})$$

The formula for area at  $n+1$ th stage will be –



$$\begin{aligned} A_{n+1} &= A_n + (3^{(3/2)-2(n+1)} * 4^{(n+1)-2}) \text{ ----- [1]} \\ &= A_n + (3^{(3/2)-2n-2} * 4^{n-1}) \end{aligned}$$

Now, let the number of additional triangles at each stage be  $U_n$ .

Thus, the general formula for area becomes –

$$A_n = A_{n-1} + U_n$$

Let it be true for  $n = k$ .

Therefore, 
$$A_k = A_{k-1} + U_k$$

To be proved for  $n = k+1$

Therefore, 
$$A_{k+1} = A_k + U_{k+1}$$

RHS

$$= A_k + U_{k+1}$$

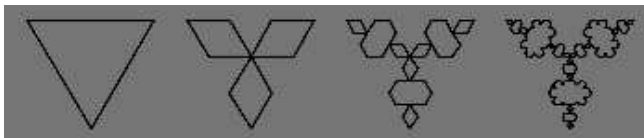
From equation 1, we get

$$\begin{aligned} &= A_k + (3^{(3/2)-2(n+1)} * 4^{(n+1)-2}) \\ &= A_k + (3^{(3/2)-2n-2} * 4^{n-1}) = \underline{\text{LHS}} \end{aligned}$$

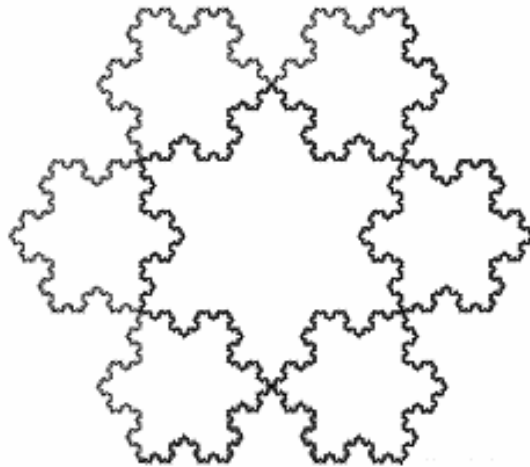
Thus, I have proved my general formula for the area of the Koch snowflake through induction.

## CONCLUSION

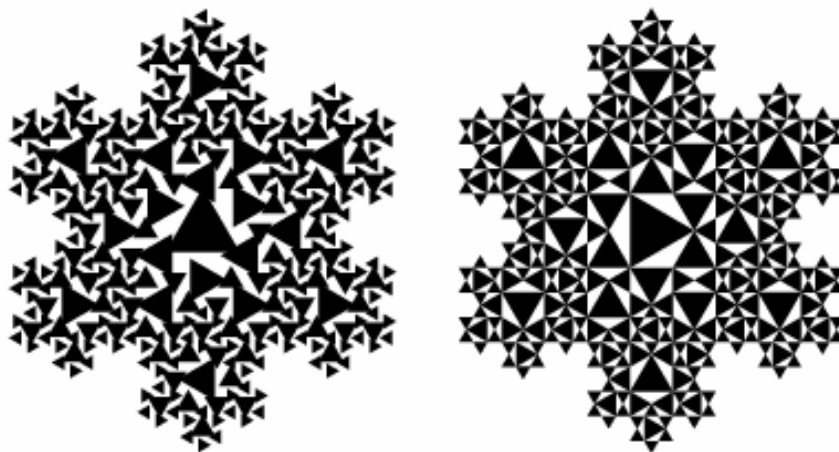
Now that I have finished this extensively researched portfolio, I have actually got inspired to know more about this project of mine and I am sure that you will also, after reading my portfolio, get the surge to know more about the Koch snowflake. After finishing this portfolio I was so inspired that I searched on the internet for figures similar to the Koch Snowflake and have found several figures such as the **Koch Antisnowflake**, the **Exterior Snowflake**, the **Pentaflake** and the **Sierpinski Curve**. Finally, I hope that you like my portfolio and enjoy reading it.



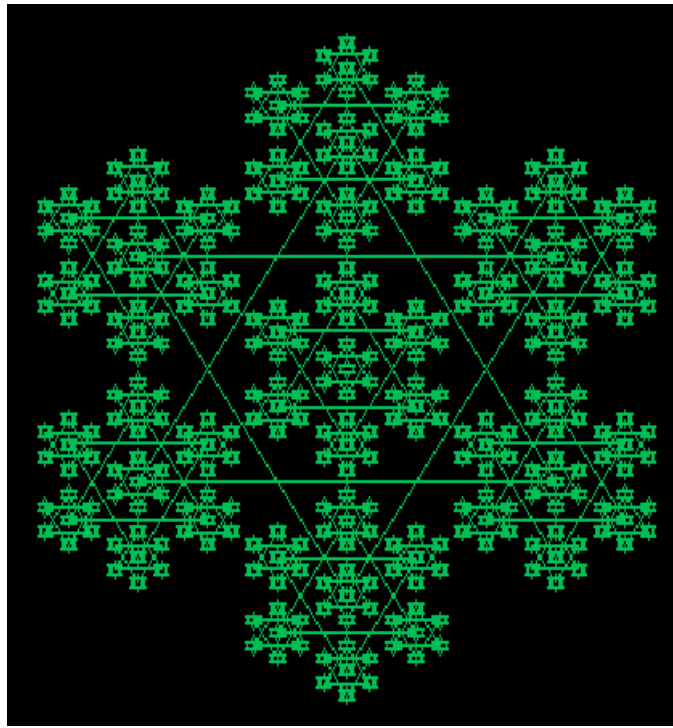
*Stages in Koch Antisnowflake*



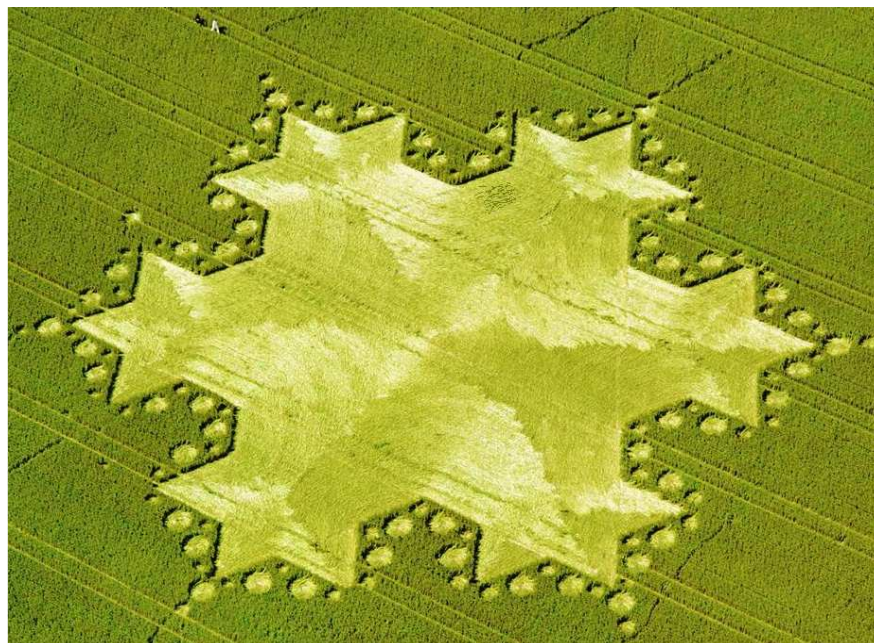
Some beautiful tillings can be made with iterations towards the Koch Snowflake.



Some beautiful modifications of the Koch Snowflake involving the constituent triangles with filled-in triangles inscribed, rotated at some angles.



An extraordinary Koch Snowflake created in LOGO.



Koch Snowflake – Silbury Hill, England  
**A Beauty Indeed!**

**REFERENCES:-**

<http://www.nonoscience.info/2006/08/28/the-koch-curve-and-visual-resolution> - 12/7/2007

<http://www.mathworld.wolfram.com/KochSnowflake.html> - 12/7/2007

<http://ecademy.agnesscott.edu/~lriddle/ifs/ksnow/ksnow.htm> - 20/7/2007

<http://www.math.ubc.ca/~cass/courses/m308-05b/projects/fung/page.html> - 20/7/2007

<http://math.arizona.edu/~caine/chaos.html> - 25/7/2007

<http://mathworld.wolfram.com/KochAntisnowflake.html> - 30/7/2007

[http://jeffreymunro.com/amazingpics/index\\_old.php?showimage=318](http://jeffreymunro.com/amazingpics/index_old.php?showimage=318) - 1/8/2007

<http://www.seasip.demon.co.uk/Unix/Joyce/screens.html> - 1/8/2007