Mathematics HL Portfolio

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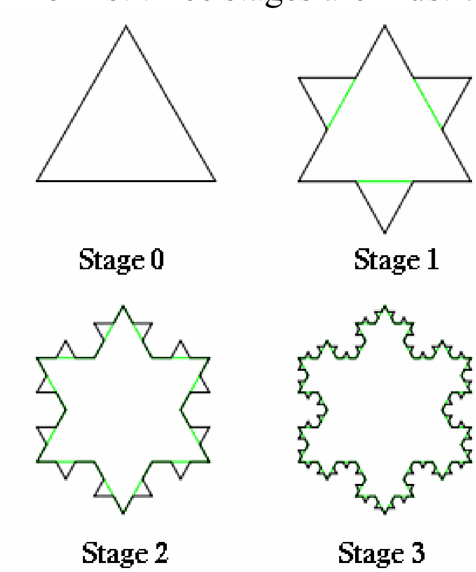
Class 12 "IB" (C).

The Koch snowflake is also known as the Koch island, which was first described by Helge von Koch in 1904. Its building starts with an equilateral triangle, removing the inner third of each side, building another equilateral triangle with no base at the location where the side was removed, and then repeating the process indefinitely.

The first three stages are illustrated in the figure below



Each step in the process is the repeating of the previous step hence it is called iteration.

If we let N_n = the number of sides, l_n = the length of a single side, P_n = the length of the perimeter, and A_n = the area of the snowflake, all at n^{th} stage, we shall get the following table for the 1^{st} three iterations.

Table no.1: the value of N_n, l_n, P_n, and A_n, at the stage zero and the following three stages.

n	N_n	l_n	P_n	A_n
0	3	1	3	$(\sqrt{3})/4$
1	12	1/3	4	$(\sqrt{3})/3$
2	48	1/9	48/9	$10(\sqrt{3})/27$
3	192	1/27	192/27	$94(\sqrt{3})/243$

Note: assume that the initial side length is 1.

We can see from the above table that the number of sides isn multiplied by four at each iteration. The length of each side is divided by 3 in each step, thus it is 1/3 the length of the same side in the previous step. As for the perimeter, the perimeter equals the number of sides multiplied by the length of a single side, hence that we have the above values where that $P_n = N_n \times l_n$. As for the area of the diagram, it is equal to the area of the original triangle plus the area of the new smaller triangles added in each step, and since it's an equilateral triangle its height (the original triangle) is equal to $(\sqrt{3})/2$, which was found from the fact that the angels of the equilateral triangle are 60^o, and using a segment which bisected the base and was normal to it. So using known famous triangles which the angles 30^o, 60^o, and 90^o

** Each graph of every single set of values plotted against the value of n, are in the following pages respectively.

Number of Sides (N_n)

One side of the figure from the previous stage becomes four sides in the following step, thus we begin with three sides, and the general expression for the number of sides in the Koch Snowflake will be:

$N_n = 3(4)^n$
At the nth stage.

For iterations 0, 1, 2 and 3, the number of sides is 3, 12, 48 and 192, respectively.

Or it can be derived from the value of N_n from the zero stage till the third stage

$N_0= 3,\quad N_1= 12,\quad N_2=48,\quad N_3=192$
$N_0= 3(1),\quad N_1= 3(4),\quad N_2= 3(16),\quad N_3 = 3(64)$
$N_0= 3(4)^0,\quad N_1=3(4)^1,\quad N_2=3(4)^2,\quad N_3= 3(4)^3$

Hence, $N_n= 3(4)^n$

Length of a single side (l_n)

As we proceed in each stage the length of any side is 1/3 the length of the side from the preceding stage. If we begin with an equilateral triangle with side length 1, then the length of a side in n^{th} iterations is

$l_n= 1/ (3)^n$

For stage 0 to 3, l_n = 1, 1/3, 1/9 and 1/27.

Or as in the above it can from the value of l_n from the zero stage till the third stage

l_0=1, l_1=1/3, l_2= 1/9, l_3= 1/27

l_0=1/ $(3)^0$, l_1=1/ $(3)^1$, l_2= 1/ $(3)^2$, l_3= 1/ $(3)^3$

And hence $l_n = 1/ (3)^n$

Perimeter (P_n)

Since the lengths of every side in every iteration of the Koch Snowflake are the same, then perimeter is simply the number of sides multiplied by the length of a side
$P_n = (N_n) (l_n)$
$P_n = (3(4)^n) (1/ (3)^n)$
For the nth stage.

Again, for the first 4 steps (0 to 3) the perimeter is 3, 4, 16/3, and 64/9.
As we can see, the perimeter increases by 4/3 times each iteration so we can rewrite the formula as
$P_n = 3(4/3)^n$

Or it can be derived from the value of P_n from the zero stage till the third stage

P_0= 3, P_1= 4, P_2= 48/9, P_3= 192/27
P_0= 3, P_1= 3(4/3), P_2= 3 (16/9), P_3= 3 (64/27)
P_0= 3 $(4/3)^0$, P_1= $3(4/3)^1$, P_2= $3(4/3)^2$, P_3= $3(4/3)^3$

And hence P_n= $3(4/3)^n$

The area of the snowflake (A_n)

In each iteration we take the area of the previous snowflake or shape, and then add the area of all of the new smaller triangles added.

And the general expression can be derived from the values of A_n from the zero stage till the third stage

$A_0 = (\sqrt{3})/4$,

$A_1 = (\sqrt{3})/3$, $A_1 = [(\sqrt{3})/4] + (1/3)((\sqrt{3})/4)$, $A_1 = [(\sqrt{3})/4] + (4/9)^0(1/3)((\sqrt{3})/4)$

$A_2 = 10(\sqrt{3})/27$, $A_2 = [(\sqrt{3})/3] + ((\sqrt{3})/27)$, $A_2 = [(\sqrt{3})/3] + (1/3)(4/9)^1((\sqrt{3})/4)$

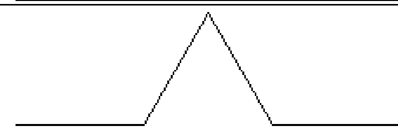
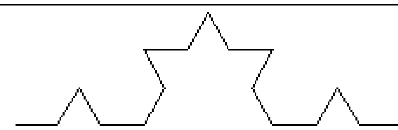
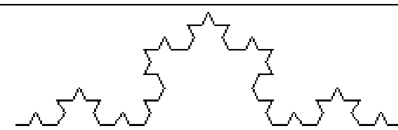
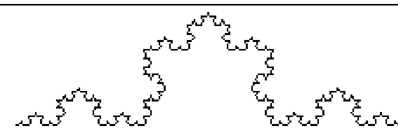
$A_3 = 94(\sqrt{3})/243$, $A_3 = [10(\sqrt{3})/27] + (4(\sqrt{3})/243)$, $A_3 = [10(\sqrt{3})/27] + (1/3)(4/9)^2((\sqrt{3})/4)$

And hence we can see that the general expression is

$A_n = A_{n-1} + (1/3)(4/9)^{n-1}((\sqrt{3})/4)$

From the general expressions above we can find N_n, l_n, P_n, and A_n, at n=4.

Where that $N_4 = 3(4)^4 = 768$, $l_4 = 1/(3)^4 = 1/81$,
$P_n = 3(4/3)^4 = 256/27$,
$A_4 = A_{4-1} + (1/3)(4/9)^{4-1}((\sqrt{3})/4) = A_3 + (1/3)(4/9)^3((\sqrt{3})/4)$
$= 0.654172$

0	
1	
2	
3	
4	

One side of the triangle through step 0 till step 4.

As n increases the area converges to a certain value where that the 1st six decimals are equal in the successive terms, and such patterns beings to appear at stage number seventeen where that A_{17}= 0.691693219 and A_{18}= 0.691693719, so we can see that $A_{n+1} = A_n$ to six places of decimals at n=17. For the other values as n get larger there is no value of n where that $l_n = l_{n+1}$ to six places of decimals.
The perimeter as n $\rightarrow \infty$ the perimeter becomes very large going to infinity, and the area converges to 0.692820323 which is equal almost to (8/5) (($\sqrt{3}$)/4) which is (8/5) the area of the original triangle at step 0.

Hereby the general expression for A_n will be proved by induction

$A_0= (\sqrt{3})/4$, $A_1= (\sqrt{3})/3$

$A_n = A_{n-1} + (1/3)\ (4/9)^{\ n-1}((\sqrt{3})/4)$ prove it is true for n=1

$A_1 = A_{1-1} + (1/3)\ (4/9)^{\ 1-1}((\sqrt{3})/4)$
$A_1 = A_0 + (1/3)\ (4/9)^0((\sqrt{3})/4)$
$A_1 = (\sqrt{3})/4 + ((\sqrt{3})/12)$
$(\sqrt{3})/3 = (4(\sqrt{3})/12) = (\sqrt{3})/3$
Then Left hand side equals Right hand Side

Now assume it is true for n=k that is
$A_k = A_{k-1} + (1/3)\ (4/9)^{\ k-1}((\sqrt{3})/4)$
Now we want to prove it is true for n = k+1
$A_{k+1} = A_{\ (k+1)-1} + (1/3)\ (4/9)\ (^{k+1)-1}((\sqrt{3})/4)$
$A_{k+1} = A_{\ k} + (1/3)\ (4/9)^{\ k}((\sqrt{3})/4)$
$A_1 = A_0 + (1/3)\ (4/9)^0((\sqrt{3})/4)$
$(\sqrt{3})/3 = [(\sqrt{3})/4] + (1/3)\ ((\sqrt{3})/4) = (4(\sqrt{3})/12)$
$= (\sqrt{3})/3$

Since A_n is true for n=1, so it true for n=k, then it true for n= k+1, then is true for all values of n