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Calculus AB/BC
7th & 8th
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Investigating Ratios of Areas and Volumes

1.Given the function y= x^2, consider the region formed by this function from x=0 to x=1 and the x-axis. Label this area B. Label the region from y=0 to y=1 and the y-axis area A.

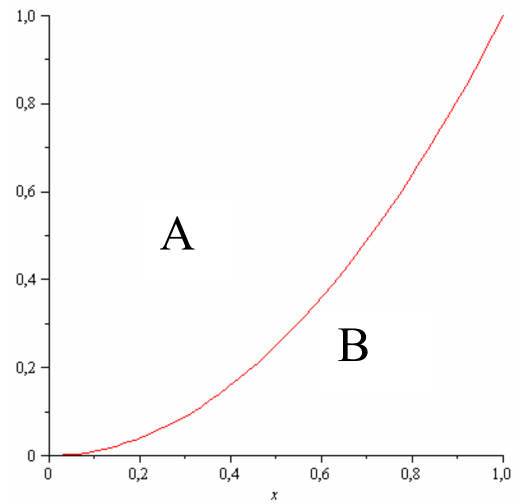

Figure 1.0

A. Find the ratio of area A: area B.

The area of A for the given function of $y = x^2$ is $\frac{2}{3}$.

Area A $\int_0^1 1 - x^2 dx = \frac{2}{3}$

The area of B for the given function of $y = x^2$ is $\frac{1}{3}$

Area B: $\int_0^1 x^2 dx = \frac{1}{3}$

Thus the ratio of area A : area B can be given as 2:1.

B. Calculate the ratio of the areas for other functions of the type $y = x^n, n \in Z^+$ between $x = 0$ and $x = 1$. Make a conjecture and test your conjecture for other subsets of the real numbers.

Area A: 1-$\frac{1}{n+1} = \frac{n}{n+1}$

Area B: $\frac{1}{n+1}$

Conjecture: Given the function $y = x^n, n \in Z^+$ the area of A in ratio of the area of B can be given as n:1.

Other Tests for the equation $y = x^n$ $[0,1]$

Equation	Area of A	Area of B	Ratio
$y = x^3$	$\frac{3}{4}$	$\frac{1}{4}$	3:1
$y = x^4$	$\frac{4}{5}$	$\frac{1}{5}$	4:1
$y = x^5$	$\frac{5}{6}$	$\frac{1}{6}$	5:1
$y = x^6$	$\frac{6}{7}$	$\frac{1}{7}$	6:1

2. Does your conjecture hold only for areas between x=0 and x=1?
Examine for x=0 and x=2; x =1 and x =2 etc.

Ex 1) The area of A for the function y=x^2 [0,2] is 8

Area A: $\int_0^2 4 - x^2 dx = \frac{16}{3}$

The area of B for the function y=x^2 [0,2] is 16/3

Area B: $\int_0^2 x^2 dx = \frac{8}{3}$

Explanation: This supports my conjecture that the area of A in ratio to the area of B is n:1. In the equation $y = x^2$, the value for n is 2, thus the ratio of the area of A, $\frac{16}{3}$, to the area of B, $\frac{8}{3}$, is 2:1.

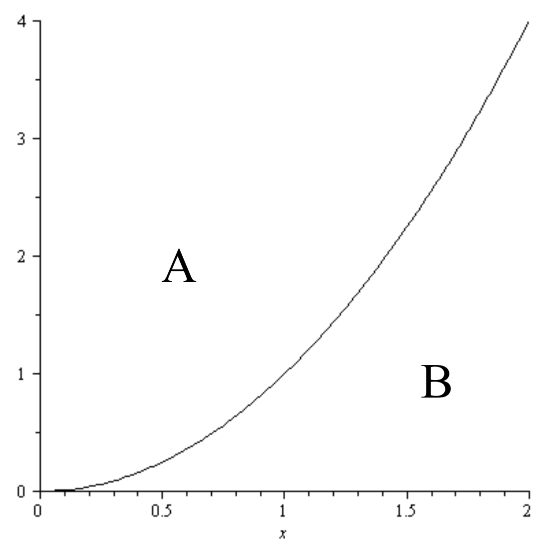

Figure 1.2

Ex 2) The area of A_1 for the function $y = x^2$ $[1,2]$ is

Area A: $\int_1^2 4 - x^2 dx = \frac{5}{3}$

The area of B for the function $y = x^2[1,2]$ is $\frac{7}{3}$

Area B: $\int_1^2 x^2 dx = \frac{7}{3}$

Explanation: The area of B would be known as $\frac{7}{3}$ and the area of A_1would be $\frac{5}{3}$. A_2 can simply be found by drawing a rectangle to make a 1x3 rectangle, as in Figure 1.3 which was made using Maple. The area of A would thus be $A_1 + A_2$ which would give you $\frac{14}{3}$. This would prove my conjecture since the area of A to the area of B is 2:1 which is the same as my conjecture n:1.

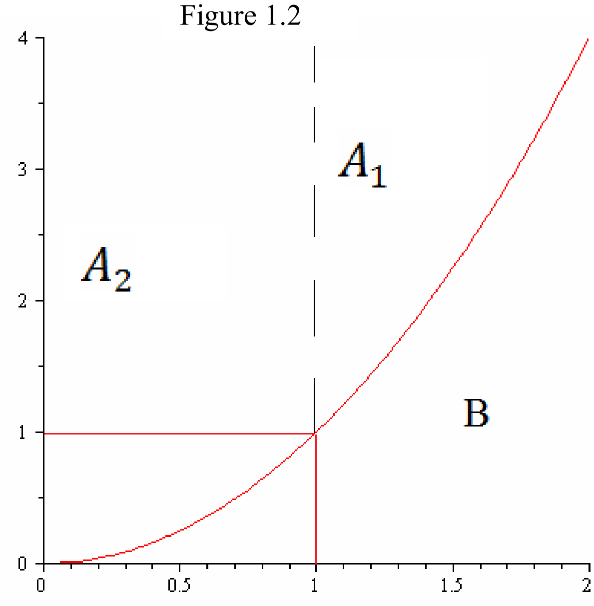

Figure 1.3

Ex 3) $y = x^3$ $[0,3]$

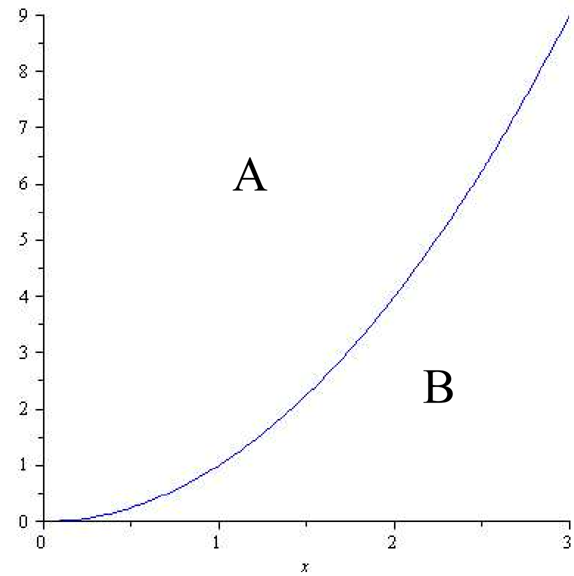

Figure 1.4

Area A: $\int_0^3 9 - x^2 dx = 18$

Area B: $\int_0^3 x^2 dx = 9$

Explanation: The area of A can simply be found by taking the integral as shown above. The area of B can also be done by taking the integral as shown above. The ratio of the area of A to the area of B is 18:9 which can be simplified to 2:1. This supports my conjecture that the ratio of area A : area B is n:1.

3. Is your conjecture true for the general case $y = x^n$ from $x = a$ to $x = b$ such that $a < b$ and for the regions defined below? If so prove it; if not explain why not.
Area A: $y = x^n, y = a^n, y = b^n$ and the y-axis
Area B: $y = x^n, x = a, x = b$ and the x-axis

Area A: $\int_{a^n}^{b^n} y^{\frac{1}{n}}\, dy = \frac{n}{n+1} \times y^{\frac{n+1}{n}} \Big|_{a^n}^{b^n} = \frac{nb^{n+1}}{n+1} - \frac{na^{n+1}}{n+1} = \frac{n(b^{n+1} - a^{n+1})}{n+1}$

Explanation: The equation of $y = x^n$ can be stated in terms of y as $x = \sqrt[n]{y}$ or $x = y^{\frac{1}{n}}$. The integral for the area of A is put into terms of y and then solved.

Area B: $\int_a^b x^n dx = \frac{x^{n+1}}{n+1} \Big|_{a^n}^{b^n} = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} = \frac{(b^{n+1} - a^{n+1})}{n+1}$

Explanation: Instead of putting the integral in terms of y, we put the integral in terms of x and then we solve.

Since my conjecture was n:1 which is the same as $\frac{n}{1}$. So we can state the ratio as $\frac{area\ A}{area\ B}$ which would be $\frac{\frac{n(b^{n+1} - a^{n+1})}{n+1}}{\frac{(b^{n+1} - a^{n+1})}{n+1}}$. This would simplify to $\frac{n}{1}$ which would support my conjecture.

4. Are there general formulae for the ratios of the volumes of revolution generated by the regions A and B when they are each rotated about:

(a) x-axis

(b) y-axis

State and prove your conjecture.

<u>Region A around x-axis</u>

Part A. $\pi \int_0^a (b^n)^2 - (a^n)^2 dx = xb^{2n} - xa^{2n} \Big|_0^{a^n} = \pi(ab^{2n} - a^{2n+1})$

Part B. $\pi \int_a^b b^{2n} - x^{2n} dx = xb^{2n} - \frac{x^{2n+1}}{2n+1} \Big|_a^b = \pi\left[\left(b^{2n+1} - \frac{b^{2n+1}}{2n+1}\right) - \left(ab^{2n} - \frac{a^{2n+1}}{2n+1}\right)\right]$

Volume of Region A: $\pi\left[(ab^{2n} - a^{2n+1}) + \left(b^{2n+1} - \frac{b^{2n+1}}{2n+1}\right) - \left(ab^{2n} - \frac{a^{2n+1}}{2n+1}\right)\right] =$
$\pi[(b^{2n+1} - a^{2n+1})(1 - \frac{1}{2n+1})] = [(b^{2n+1} - a^{2n+1})(\frac{2n}{2n+1})]$

Explanation: Region A is simply divided into 2 segments. One being from [0,a] and the other from [a,b]. They are both integrated separately and then added together. After some simplification we get the volume of region A.

<u>Region B around x-axis</u>

π

$\int_a^b (x^n)^2 dx = \frac{x^{2n+1}}{2n+1} \Big|_a^b = \pi\left(\frac{b^{2n+1} - a^{2n+1}}{2n+1}\right)$

Explanation: Region B has no hole in the center thus it can be integrated together without separation.

Conjecture: The ratio of volumes of Region A to Region B is 2n:1. This can be proven by taking the volume of Region A and dividing it by the volume of Region B.

$$\frac{\pi\ [(b^{2n+1} - a^{2n+1})(\frac{2n}{2n+1})]}{\pi\left(\frac{b^{2n+1} - a^{2n+1}}{2n+1}\right)} = \frac{2n}{1} = 2n:1$$

Region A around y-axis

$$\pi \int_{a^n}^{b^n} (\sqrt[n]{y})^2 dy = \frac{ny^{\frac{2+n}{n}}}{2+n}\Bigg|_{a^n}^{b^n} = \frac{nb^{2+n}}{2+n} - \frac{na^{2+n}}{2+n} = n\pi\left(\frac{b^{2+n}-a^{2+n}}{2+n}\right)$$

Explanation: Since region A around the y-axis is a solid figure there is no need to have multiple integrals. So the volume of region A is simply found by using the disk method.

Region B around y-axis

$$2\pi\int_a^b (x)(x^n)dx = \frac{x^{n+2}}{n+2}\Bigg|_a^b = 2\pi\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)$$

Explanation: Since region B contains a portion cut out, to find the area of B we use the shell method.

Conjecture: My conjecture for the volumes of regions when rotated about the y-axis is the ratio of n:2. This can be proven by taking region A and dividing it by region B.

$$\frac{n\pi\left(\frac{b^{2+n}-a^{2+n}}{2+n}\right)}{2\pi\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)} = \frac{n}{2}$$