Introduction

In this portfolio, I will investigate the ratio of areas formed when $y = x^n$ is graphed between two arbitrary parameters $x = a$ and $x = b$ such that $a < b$.

Ratio of Area

Given the function $y = x^2$, I will first consider the region formed by this function from $x = 0$ to $x = 1$ and the x-axis. The region will be labeled B. The region from $y = 0$ to $y = 1$ and the y-axis will be labeled A. A diagram of this is show below:

Finding the ratio of area A: area B:

By integrating this function from $x = 0$ to $x = 1$, we can find the area under the curve, area B.

$$B = \int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Since the total area of area A and B is 1, we can simply subtract the area of B from 1 to get area A.

$$A = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus the ratio of area A: area B is 2:1.

Now, I will consider the ratio of areas for other functions of the type $y = x^n,\ n \in \square^{+}$ between $x = 0$ and $x = 1$.

Function	Area of B (under the curve)	Area of A	Ratio
$y = x^3$	$\int_0^1 x^3 dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$	$1 - \frac{1}{4} = \frac{3}{4}$	3:1 = 3
$y = x^4$	$\int_0^1 x^4 dx = \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5} - 0 = \frac{1}{5}$	$1 - \frac{1}{5} = \frac{4}{5}$	4:1 = 4
$y = x^5$	$\int_0^1 x^5 dx = \left[\frac{1}{6}x^6\right]_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$	$1 - \frac{1}{6} = \frac{5}{6}$	5:1 = 5

The graphs of these functions are shown below:

After considering several examples of the function of the type $y = x^n,\ n \in \square^{+}$ between $x = 0$ and $x = 1$, a clear pattern emerges. The pattern for the ratio seems to be *n:1* or just *n*.

Proof of Conjecture:

$$\int_0^1 x^n dx = \left[\frac{1}{n+1}x^{n+1}\right]_0^1 = \left(\frac{1}{n+1}1^{n+1}\right) - \left(\frac{1}{n+1}0^{n+1}\right)$$

$$B = \frac{1}{n+1}$$

$$A = 1 - \frac{1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} = \frac{n}{n+1}$$

$$A : B = \frac{n}{n+1} : \frac{1}{n+1} = n:1 = n$$

To test the conjecture, I will now consider other values of *n* by including other subsets of real numbers. The chart below shows functions of the type $y = x^n,\ n \in$ between $x = 0$ and $x = 1$. The results below show that the conjecture holds for other subsets of real numbers.

Function	Area of B (under the curve)	Area of A	Ratio
$y = x^0$	$\int_0^1 x^0 dx = 1$	0	0:1 = 0
$y = x^{1/2}$	$\int_0^1 x^{1/2} dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$	$1 - \frac{2}{3} = \frac{1}{3}$	1:2 = ½
$y = x^{5/3}$	$\int_0^1 x^{5/3} dx = \left[\frac{3}{8}x^{8/3}\right]_0^1 = \frac{3}{8} - 0 = \frac{3}{8}$	$1 - \frac{3}{8} = \frac{5}{8}$	5:3 = 5/3

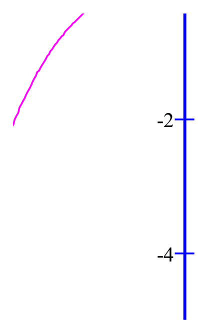
The graphs of these functions are shown below:

One important subset of real numbers has been excluded from further consideration is negative numbers. This is because no corresponding y-value exists at $x = 0$. There is a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0,$ as they approach infinity. Thus $y = x^{-n}$ cannot be integrated as infinity is not a valid area. The graph below demonstrates them phenomena for $y = x^{-2}$:

To further test the conjecture, I will now consider the areas between other x-values. I will examine if the conjecture holds for $x = 0$ and $x = 2$, $x = 1$ and $x = 2$, and $x = 2$ and $x = 5$.

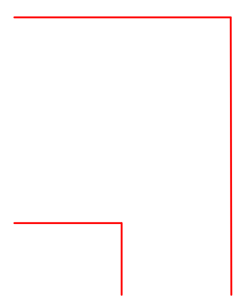
First, I will consider functions of the type $y = x^n$, $n \in$ □ between $x = 0$ and $x = 2$. The conjecture seems to hold for the bounds $x = 0$ and $x = 2$.

Function	Area of B (under the curve)	Area of A	Ratio
$y = x^0$	$\int_0^2 x^0 dx = 2$	0	0:2 = 0
$y = x^2$	$\int_0^2 x^2 dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$	$8 - \frac{8}{3} = \frac{16}{3}$	2
$y = x^{2/3}$	$\int_0^2 x^{2/3} dx = \left[\frac{3}{5}x^{5/3}\right]_0^2 = 1.9048\ldots - 0 = 1.9048\ldots$	$2\sqrt[3]{4} - 1.9038\ldots = 1.2699\ldots$	1.2699…/ 1.9038… = 2/3



Area of A and B is now defined as the following:
Area A: $y = x^n$, $y = a^n$, $y = b^n$, and the y-axis
Area B: $y = x$, $x = a$, $y = b$, and the x-axis

The graph below is an example of the bounds for $y = x^2$ from $x=1$ to $x=2$



Next, I will consider functions of the type $y = x^n,\ n \in \mathbb{R}$ between $x = 1$ and $x = 2$. The conjecture seems to hold for the bounds $x = 1$ and $x = 2$.

Function	Area of B (under the curve)	Area of A	Ratio
$y = x^0$	$\int_1^2 x^0 dx = 1$	0	0:1 = 0
$y = x^2$	$\int_1^2 x^2 dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$	$7 - \frac{7}{3} = \frac{14}{3}$	14:7 = 2
$y = x^{2/3}$	$\int_1^2 x^{2/3} dx = \left[\frac{3}{5}x^{5/3}\right]_1^2 = 1.9048\ldots - \frac{3}{5} = 1.3048\ldots$	$(2\sqrt[3]{4} - 1) -$ 1.3048…= 0.8699…	0.869…/ 1.3048… = 2/3

Finally, I will consider functions of the type $y = x^n,\ n \in \mathbb{R}$ between $x = 2$ and $x = 5$. The conjecture seems to hold for the bounds $x = 2$ and $x = 5$.

Function	Area of B (under the curve)	Area of A	Ratio
$y = x^0$	$\int_2^5 x^0 dx = 3$	0	0:3 = 0
$y = x^2$	$\int_2^5 x^2 dx = \left[\frac{1}{3}x^3\right]_2^5 = \frac{125}{3} - \frac{8}{3} = 39$	117 − 39 = 78	78:39 = 2
$y = x^{2/3}$	$\int_2^5 x^{2/3} dx = \left[\frac{3}{5}x^{5/3}\right]_2^5 =$ 8.7720…− 1.9048… = 6.8671…	$(5 * 5^{2/3} - 2\sqrt[3]{4}) -$ 6.8671…= 4.5781…	4.5781…/ 6.8671… = 2/3

After testing the conjecture for $y = x^n$ for different types of real numbers and different bounds, the conjecture still holds. Now, I will consider the conjecture in more general terms. For the general case $y = x^n$ from $x = a$ to $x = b$ such that a < b and for the regions defined below:
Area A: $y = x^n$, $y = a^n$, $y = b^n$, and the y-axis
Area B: $y = x$, $x = a$, $y = b$, and the x-axis

Proof:

$$B = \int_a^b x^n dx = \left[\frac{1}{n+1}x^{n+1}\right]_a^b = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

The formula for area A can be defined as: $\int_{f(a)}^{f(b)} f^{-1}(y)dy$

$$\int_{a^n}^{b^n} y^{\frac{1}{n}} dy = \left[\frac{y^{\left(\frac{1}{n}+1\right)n}}{\left(\frac{1}{n}+1\right)n}\right]_{a^n}^{b^n} = \left[\frac{n * y^{\frac{1}{n}} * y}{1+n}\right]_{a^n}^{b^n} = \frac{n * b * b^n}{1+n} - \frac{n * a * a^n}{1+n} = \frac{n\left(b^{n+1} - a^{n+1}\right)}{1+n}$$

Thus the ratio of A: B is n:

$$\frac{A}{B}=\frac{n\left(b^{n+1}-a^{n+1}\right)(1+n)}{1+n\left(b^{n+1}-a^{n+1}\right)}=n$$

Ratios of Volume

After considering and proving that the ratio of the area A: area B is simply n, I will now consider a general formula for the ratios of the volumes of revolution generated by the regions A and B when they are rotated about the x-axis and y-axis.

Proof for x-axis conjecture:

I will use the disc method to solve for volumes of revolution: V = $\pi\int_a^b y^2dx$, y is the function $y = x^n$

$$V_B = \pi\int_a^b y^2dx = \pi\int_a^b \left(x^n\right)^2dx = \pi\int_a^b x^{2n}dx = \pi\int_a^b x^{2n}dx = \pi\frac{1}{2n+1}\left(b^{2n+1}-a^{2n+1}\right)$$

$$V_A = \pi\int_a^b \left(b^n\right)^2-\left(x^n\right)^2dx = \pi\left[b^{2n}x-\frac{1}{2n+1}x^{2n+1}\right]_a^b = \pi\left[b^{2n}(b-a)-\frac{1}{2n+1}(b^{2n+1}-a^{2n+1})\right]$$

Thus $$\frac{V_A}{V_B}=\frac{\pi\left[b^{2n}(b-a)-\frac{1}{2n+1}(b^{2n+1}-a^{2n+1})\right]}{\pi\frac{1}{2n+1}(b^{2n+1}-a^{2n+1})}=\frac{b^m(b-a)}{\pi\frac{1}{2n+1}(b^{2n+1}-a^{2n+1})}+1=\frac{b^{2n}(b-a)(m+1)}{(b^{m+1}-a^{m+1})}-1$$

Proof for the y-axis:

$$V_A = \pi\int_{a^n}^{b^n} x^2dy = \pi\int_{a^n}^{b^n}\left(y^{\frac{1}{n}}\right)^2dy = \pi\int_{a^n}^{b^n}\left(y^{\frac{2}{n}}\right)dy = \pi\left[\frac{y^{\frac{2}{n}+1}}{\frac{2}{n}+1}\right]_{a^n}^{b^n} = \frac{\pi}{\frac{2}{n}+1}\left[(b^n)^{(\frac{2}{n}+1)}-(a^n)^{(\frac{2}{n}+1)}\right]$$

$$=\frac{\pi}{\frac{2}{n}+1}\left[b^{n+2}-a^{n+2}\right]$$

$$V_B = \pi\int_{a^n}^{b^n}(b^2-x^2)dy = \pi\int_{a^n}^{b^n}(b^2-y^{\frac{2}{n}})dy = \pi\left[b^2y-\frac{1}{\frac{2}{n}+1}y^{\frac{2}{n}+1}\right]_{a^n}^{b^n} =$$

$$\pi\left[b^2(b^n-a^n)-\frac{1}{\frac{2}{n}+1}(b^{n+2}-a^{n+2})\right]$$

$$\frac{V_A}{V_B}=\frac{\dfrac{\pi}{\dfrac{2}{n}+1}(b^{n+2}-a^{n+2})}{\pi\left[b^2(b^n-a^n)-\dfrac{1}{\dfrac{2}{n}+1}(b^{n+2}-a^{n+2})\right]}=\frac{\dfrac{n}{n+2}(b^{n+2}-a^{n+2})}{b^2(b^n-a^n)-\dfrac{n}{n+2}(b^{n+2}-a^{n+2})}$$

$$=\frac{n(b^{n+2}-a^{n+2})}{(n+2)b^2(b^n-a^n)-n(b^{n+2}-a^{n+2})}$$

Conclusion

Thus the ratio of area A: area B is $\frac{b^{2n}(b-a)(m+1)}{(b^{m+1}-a^{m+1})}-1$ for the x-axis and $\frac{n(b^{n+2}-a^{n+2})}{(n+2)b^2(b^n-a^n)-n(b^{n+2}-a^{n+2})}$ for the y-axis.