

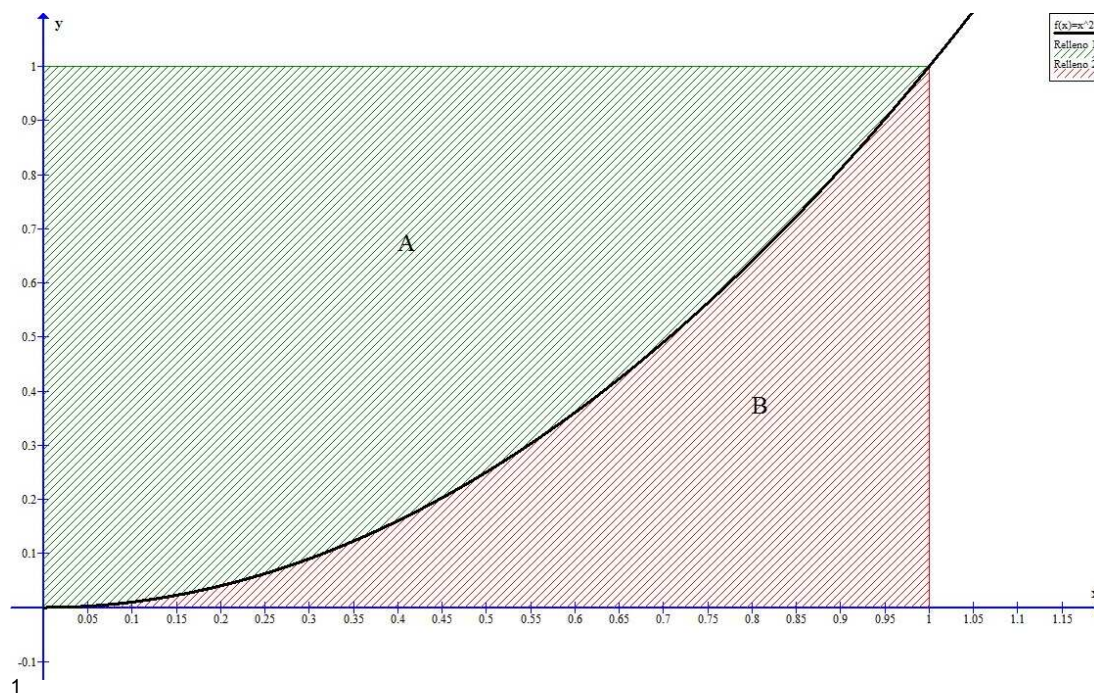
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INVESTIGATING RATIOS OF AREAS AND VOLUMES

Introduction

The objective of this portfolio assignment is to investigate the ratio of the areas formed when $y = x^n$ is graphed between arbitrary parameters $x = a$ and $x = b$ such that $a < b$. This investigation may lead to a conjecture which ends up in a general formula.

Given the function $y = x^2$, we can consider a region formed by this function from $x = 0$ and $x = 1$. The area between the function and the x-axis will be labeled B. The area from $y = 0$ to $y = 1$ and the y-axis will be labeled A.



¹ All graphs were developed on Graph software, version 4.3.

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

The formula used to find the area under a curve to the x-axis is $\int_a^b y dx$. To find the area to the y-axis the formula used is $\int_a^b x dy$.

The area between the function and the x-axis is given by:

$$B = \int_0^1 x^2 dx$$

$$B = \frac{x^3}{3} \Big|_0^1$$

$$B = \left[\frac{1}{3}(1)^3 \right] - \left[\frac{1}{3}(0)^3 \right]$$

Therefore area of B is equal to $B = \frac{1}{3}$.

So as the unit area is 1, the area of A is given by:

$$A = 1 - B$$

$$A = 1 - \frac{1}{3}$$

$$A = \frac{2}{3}$$

Another method which may be used is by getting the inverse of the function such that if $y = x^2$, the inverse would be $y = x^{\frac{1}{2}}$.

Therefore, the ratio of the areas A and B in the function $y = x^2$ is $\frac{2}{3} : \frac{1}{3}$ simplified into 2:1.

But will this ratio remain as we increase the power of the function? This will be proven by elevating the exponent of the formula by 1 each time so this can lead to a conjecture.

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

Testing variables for n

n	Function	Area	Area A	Area B	Ratio
1	$y = x$	1	$\frac{1}{2}$	$\frac{1}{2}$	1:1
2	$y = x^2$	1	$\frac{2}{3}$	$\frac{1}{3}$	2:1
3	$y = x^3$	1	$\frac{3}{4}$	$\frac{1}{4}$	3:1
4	$y = x^4$	1	$\frac{4}{5}$	$\frac{1}{5}$	4:1
5	$y = x^5$	1	$\frac{5}{6}$	$\frac{1}{6}$	5:1
N	$y = x^n$	1	$\frac{n}{n+1}$	$\frac{1}{n+1}$	$n:1$

Procedure

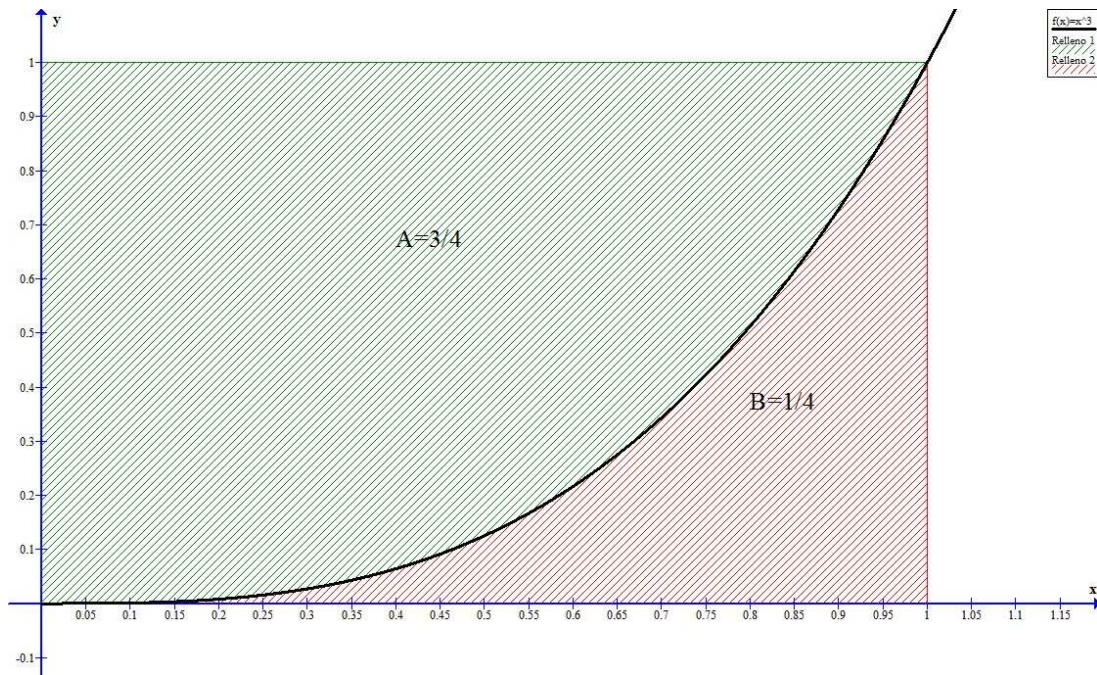
$y = x^3$	$y = x^4$	$y = x^5$	$y = x^n$
$B = \int_0^1 x^3 dx$ $B = \frac{x^4}{4} \Big _0^1$ $B = \left[\frac{(1)^4}{4} \right] - \left[\frac{(0)^4}{4} \right]$ $B = \frac{1}{4}$ $A = 1 - \frac{1}{4} = \frac{3}{4}$	$B = \int_0^1 x^4 dx$ $B = \frac{x^5}{5} \Big _0^1$ $B = \left[\frac{(1)^5}{5} \right] - \left[\frac{(0)^5}{5} \right]$ $B = \frac{1}{5}$ $A = 1 - \frac{1}{5} = \frac{4}{5}$	$B = \int_0^1 x^5 dx$ $B = \frac{x^6}{6} \Big _0^1$ $B = \left[\frac{(1)^6}{6} \right] - \left[\frac{(0)^6}{6} \right]$ $B = \frac{1}{6}$ $A = 1 - \frac{1}{6} = \frac{5}{6}$	$B = \int_0^1 x^n dx$ $B = \frac{x^{n+1}}{n+1} \Big _0^1$ $B = \left[\frac{(1)^{n+1}}{n+1} \right] - \left[\frac{(0)^{n+1}}{n+1} \right]$ $B = \frac{1}{n+1}$ $A = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

Therefore, the ratio is given by the formula $\frac{1}{n+1} : \frac{n}{n+1}$.

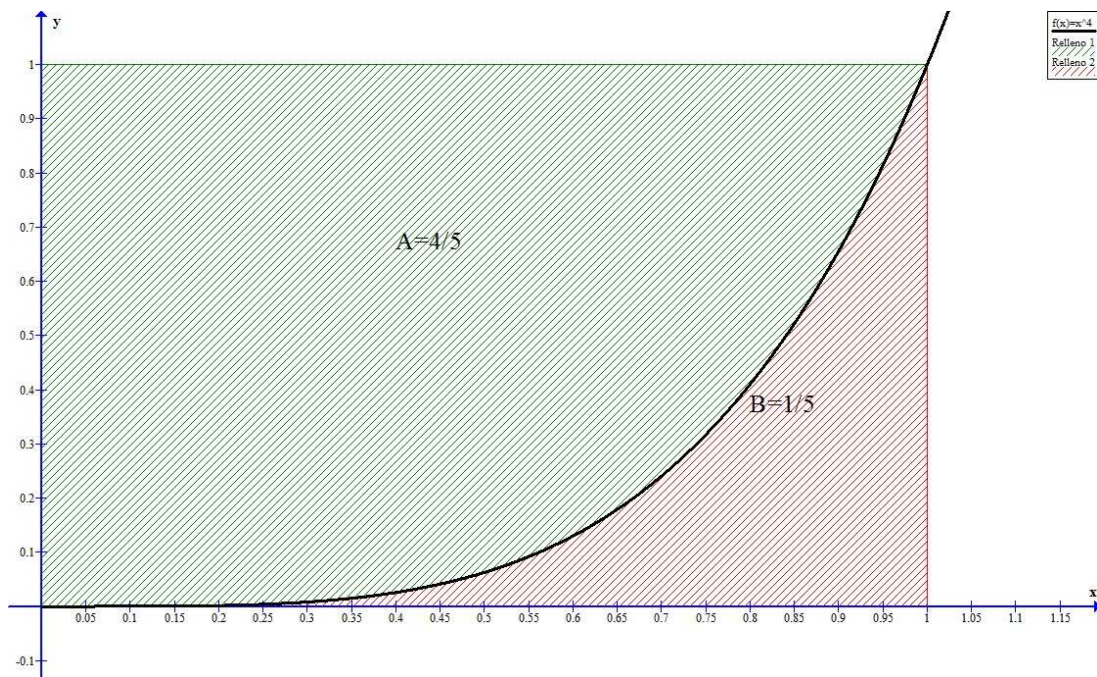
Graphing

$$y = x^3$$

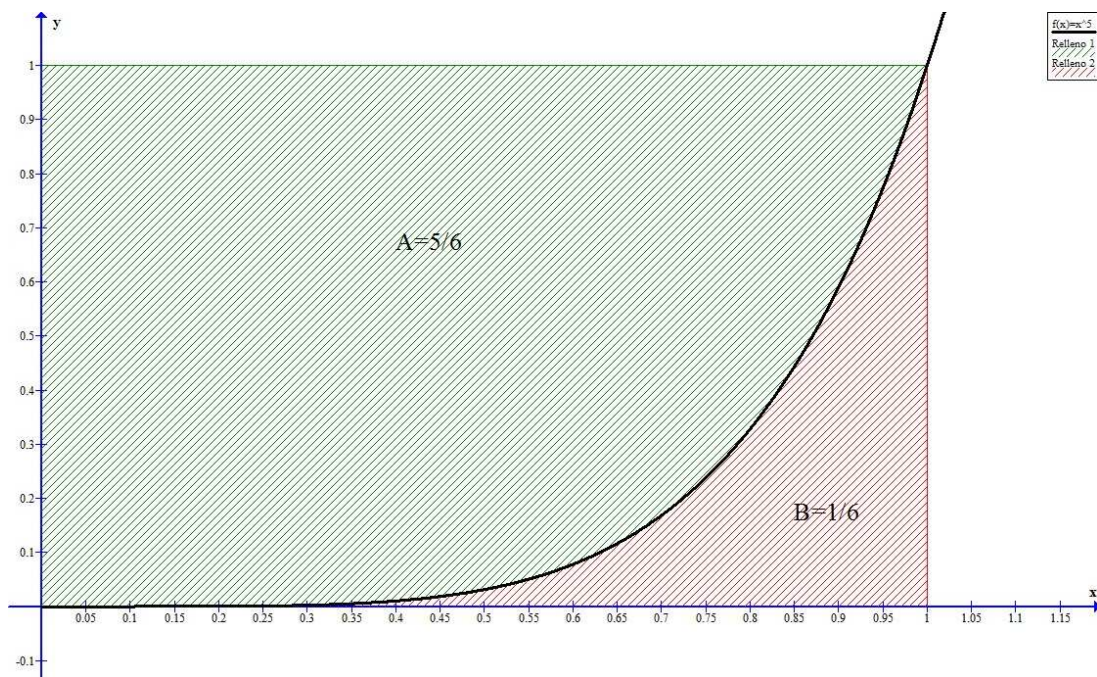


$$y = x^4$$

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009



$$y = x^5$$



Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

From the graphs we can observe that as the value of n keeps increasing the area between the curve and the x -axis gets narrower. By the other side, as n increases the area A increases too.

So as we can conclude the ratio of the areas increase proportionally to the value given to n . But what happens when the parameters of the area are changed? In this section of the investigation I will change the limits such as $x = 1$ and $x = 3$, $x = 1$ and $x = 2$, etc.

a	b	n	Area	Area A	Area B	Ratio
1	2	2	7	$\frac{14}{3}$	$\frac{7}{3}$	2:1
1	2	3	15	$\frac{45}{4}$	$\frac{15}{4}$	3:1
0	2	2	8	$\frac{16}{3}$	$\frac{8}{3}$	2:1
0	2	3	16	$\frac{48}{4}$	$\frac{16}{4}$	3:1
0	3	2	27	$\frac{54}{3}$	$\frac{27}{3}$	2:1
		n	A	$\frac{3A}{n+1}$	$\frac{A}{n+1}$	$n:1$

Procedure

Area B

$\int_1^2 x^2 dx$	$\int_1^2 x^3 dx$	$\int_0^2 x^2 dx$	$\int_0^2 x^3 dx$
$\frac{x^3}{3} \Big _1^2$	$\frac{x^4}{4} \Big _1^2$	$\frac{x^3}{3} \Big _0^2$	$\frac{x^4}{4} \Big _0^2$
$\left[\frac{(2)^3}{3} \right] - \left[\frac{(1)^3}{3} \right] = \frac{7}{3}$	$\left[\frac{(2)^4}{4} \right] - \left[\frac{(1)^4}{4} \right] = \frac{15}{4}$	$\left[\frac{(2)^3}{3} \right] - \left[\frac{(0)^3}{3} \right] = \frac{8}{3}$	$\left[\frac{(2)^4}{4} \right] - \left[\frac{(0)^4}{4} \right] = \frac{16}{4}$

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

$B = \frac{7}{3}$	$B = \frac{15}{4}$	$B = \frac{8}{3}$	$B = \frac{16}{4}$
$\int_0^3 x^2 dx$			
$\frac{x^3}{3} \Big _0^3$ $\left[\frac{(3)^3}{3} \right] - \left[\frac{(0)^3}{3} \right] = \frac{27}{3}$ $B = \frac{27}{3}$			

Area A

$\int_1^4 \frac{1}{x^2} dx$	$\int_1^4 \frac{1}{x^3} dx$	$\int_0^4 \frac{1}{x^2} dx$
$\frac{2}{3} x^{\frac{3}{2}} \Big _1^4$ $\left[\frac{2}{3} (4)^{\frac{3}{2}} \right] - \left[\frac{2}{3} (1)^{\frac{3}{2}} \right] = \frac{14}{3}$ $A = \frac{14}{3}$	$\frac{3}{4} x^{\frac{4}{3}} \Big _1^4$ $\left[\frac{3}{4} (4)^{\frac{4}{3}} \right] - \left[\frac{3}{4} (1)^{\frac{4}{3}} \right] = \frac{45}{4}$ $A = \frac{45}{4}$	$\frac{2}{3} x^{\frac{3}{2}} \Big _0^4$ $\left[\frac{2}{3} (4)^{\frac{3}{2}} \right] - \left[\frac{2}{3} (0)^{\frac{3}{2}} \right] = \frac{16}{3}$ $A = \frac{16}{3}$
$\int_0^4 \frac{1}{x^3} dx$	$\int_0^9 \frac{1}{x^2} dx$	
$\frac{3}{4} x^{\frac{4}{3}} \Big _0^4$ $\left[\frac{3}{4} (4)^{\frac{4}{3}} \right] - \left[\frac{3}{4} (0)^{\frac{4}{3}} \right] = 12$	$\frac{2}{3} x^{\frac{3}{2}} \Big _0^9$ $\left[\frac{2}{3} (9)^{\frac{3}{2}} \right] - \left[\frac{2}{3} (0)^{\frac{3}{2}} \right] = \frac{54}{3}$	

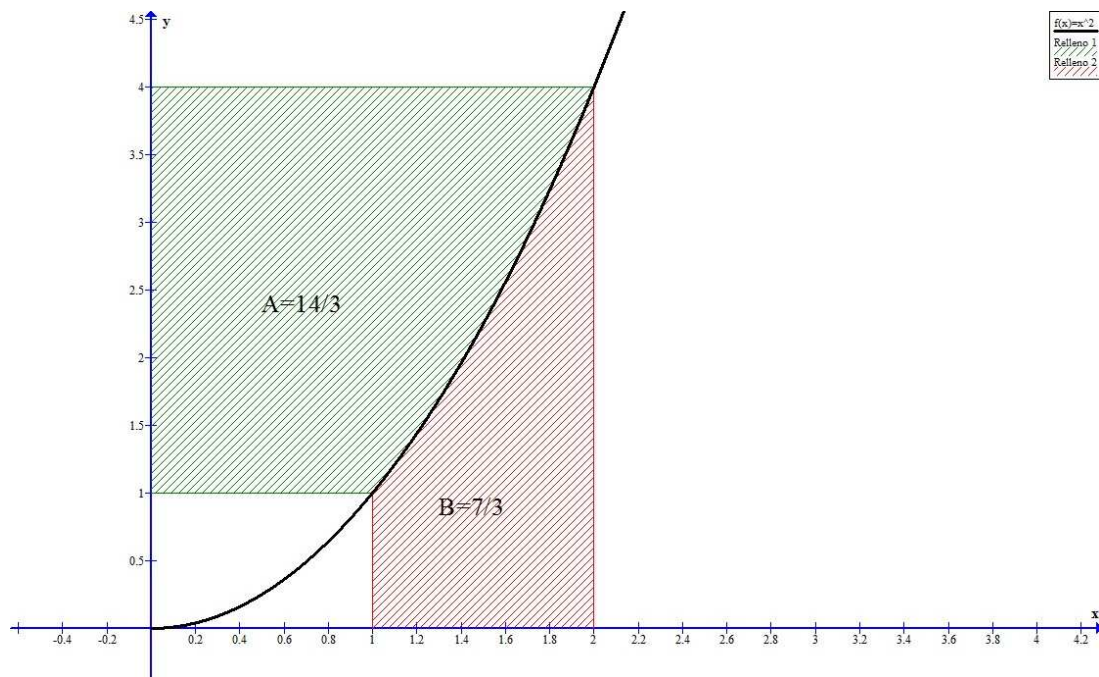
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$A = 12$	$A = \frac{54}{3}$
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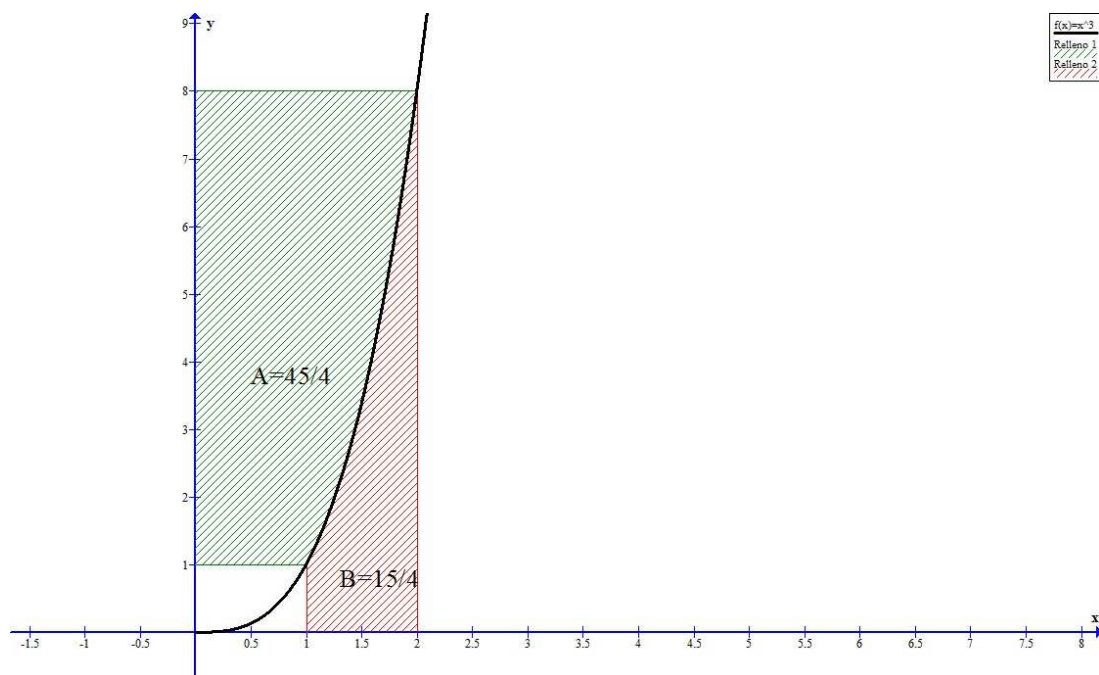
Graphing

$$y = x^2, a = 1, b = 2$$

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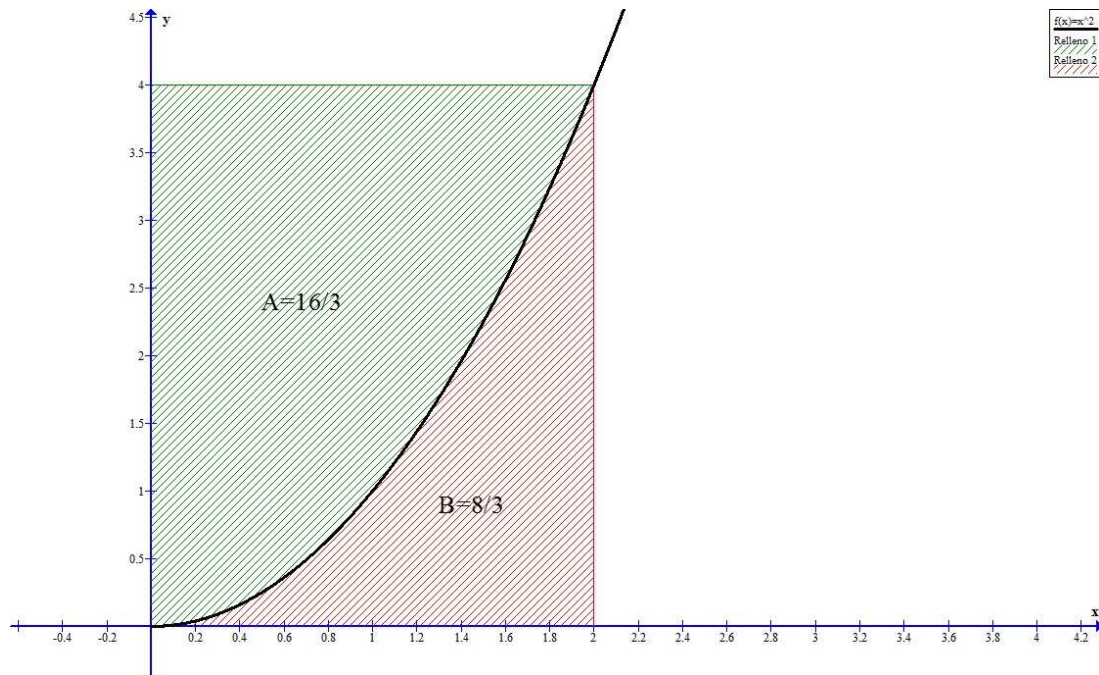


$$y = x^3, a = 1, b = 2$$

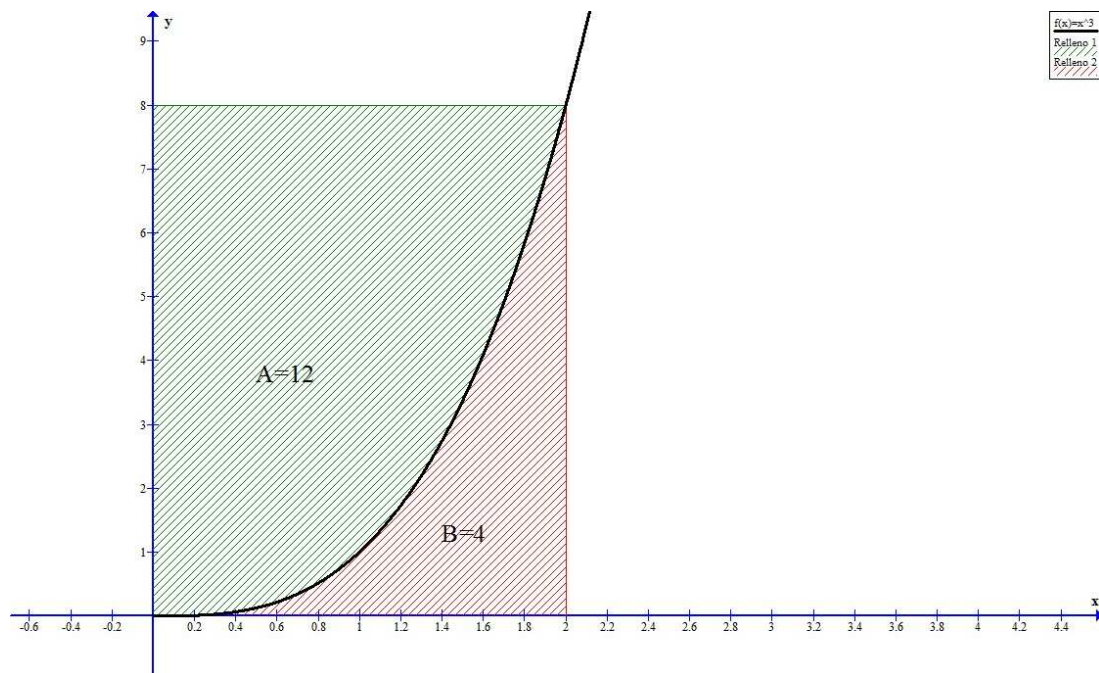


$$y = x^2, x = 0, x = 2$$

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

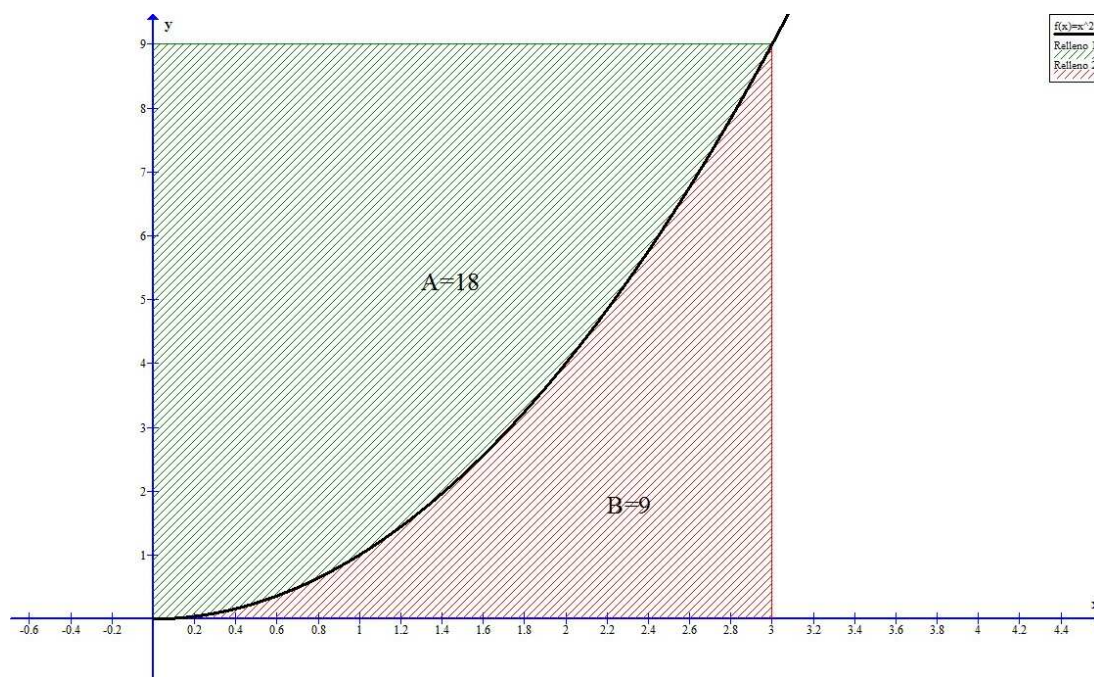


$y = x^3, x = 0, x = 2$



Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

$$y = x^2, x = 0, x = 3$$



For this section, the area of A was given by the inverse of the formula so that if $\int_1^2 x^2 dx$ is equal to B, $\int_1^4 x^{\frac{1}{2}}$ is equal to A. Therefore, based in the results of the table, the conjecture remains as n:1.

The conjecture is true for the general case $y = x^n$ from $x = a$ and $x = b$ such that $a < b$ and for the following regions:

Area A: $y = x^n$, $y = a^n$, $y = b^n$ and the y-axis.

For example, when getting the area B from $x = 1$ to $x = 2$ with the function $y = x^2$ is necessary to use $\int_1^2 x^2 dx$. So, to find the area of the curve to the y-axis (area A) it is necessary to alter the limits so that they are $y = 1$ and $y = 4$. So as $a = 1$ and $b = 2$, $a^n = 1$ and $b^n = 4$.

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

Until now the investigation has been working with ratios of areas; but what happens when dealing with volumes? In this final section, I will search for a general formula for the ratios of volumes of revolution generated by the regions A and B when they are rotated around the x-axis and the y-axis respectively.

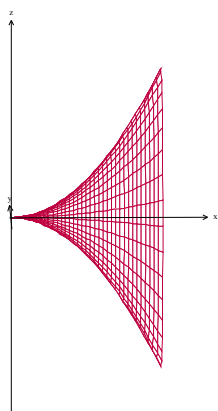
The volume of x is given by the formula $V_x = \pi \int_a^b y^2 dx$. So the formula will be used to get the area of $y = x, a = 0, b = 1$.

$$V_x = \pi \int_0^1 (x)^2 dx = \pi \left[\frac{x^3}{3} \right]_0^1 = \pi \left[\frac{(1)^3}{3} - \frac{(0)^3}{3} \right] = \frac{\pi}{3}$$

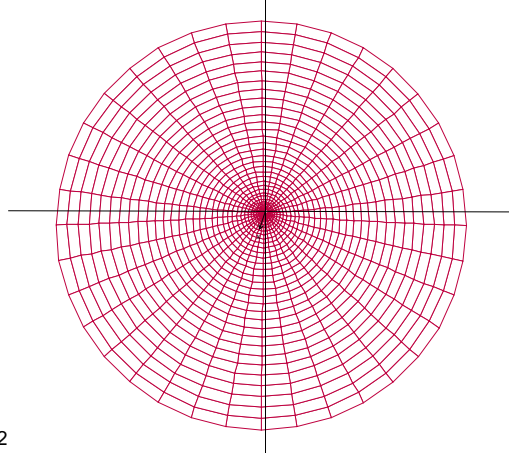
Now, I will proceed to find the volume of revolution of $y = x^2$ from $x = 0$ and $x = 1$.

$$V_x = \pi \int_0^1 (x^2)^2 dx = \pi \left[\frac{x^5}{5} \right]_0^1 = \pi \left[\frac{(1)^5}{5} - \frac{(0)^5}{5} \right] = \frac{\pi}{5}$$

gizar [y = x^2] aje axbync (0.00000,1.00000,0.00000)



gizar [y = x^2] aje axbync (0.00000,1.00000,0.00000)



2

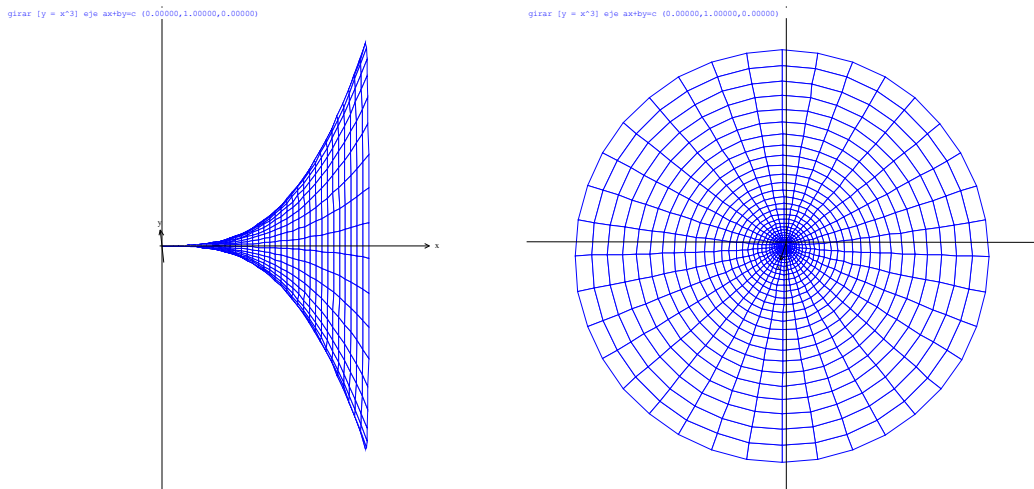
As we can see, the area under the curve rotates around the x-axis.

Now, using the same procedure I will find the volume of revolution of $y = x^3$ within the same parameters.

$$V_x = \pi \int_0^1 (x^3)^2 dx = \pi \left[\frac{x^7}{7} \right]_0^1 = \pi \left[\frac{(1)^7}{7} - \frac{(0)^7}{7} \right] = \frac{\pi}{7}$$

² Plotting for volumes of revolution was graphed on Winplot.

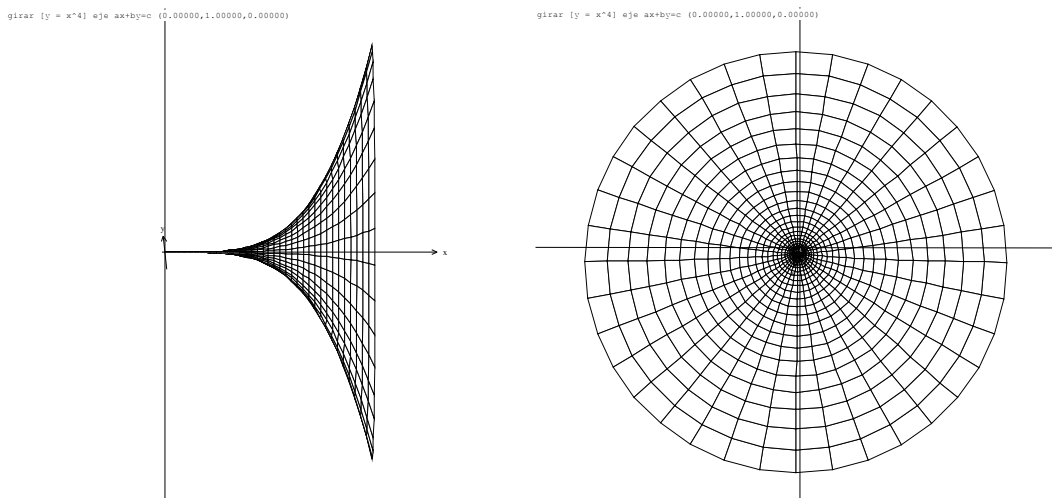
Baca Gonzalez, Gabriel Alejandro2106-005 May 2009



As we can observe, comparing this graph with the graph of $y = x^2$, the volume gets more narrow.

Now I will get the volume of revolution of $y = x^4$.

$$V_x = \pi \int_0^1 (x^4)^2 dx = \pi \left[\frac{x^9}{9} \right]_0^1 = \pi \left[\frac{(1)^9}{9} - \frac{(0)^9}{9} \right] = \frac{\pi}{9}$$



As stated in the previous graph, we can see here more clearly how the center of the volume is getting narrower as the power of the function increase.

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

Therefore a conjecture can be made, based in the results given of each calculation of the volumes, as in every increase of power in n the denominator increases by 2. The general formula for volumes of revolution in limits $x = 0$ to $x = 1$ is:

$$V_x = \frac{\pi}{2n+1}$$

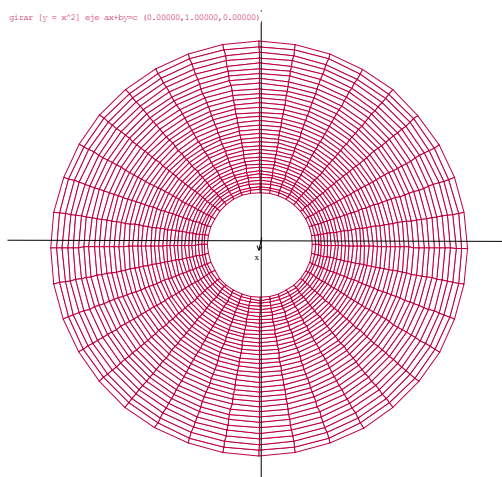
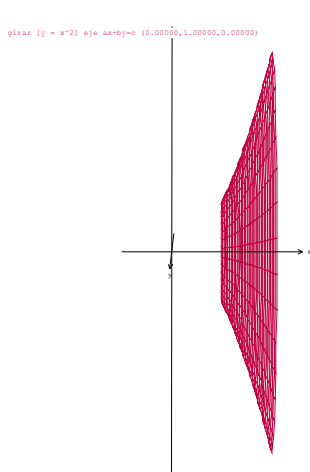
But will this conjecture apply for all volumes? Now the limits are going to be changed from $a = 0$ to $a = 1$, and from $b = 1$ to $b = 2$.

Now, for $y = x$ from $a = 1$ to $b = 2$:

$$V_x = \pi \int_1^2 (x)^2 dx = \pi \left[\frac{x^3}{3} \right]_1^2 = \pi \left[\frac{(2)^3}{3} - \frac{(1)^3}{3} \right] = \frac{7\pi}{3}$$

Now, using the same procedure I will find the volume of revolution of $y = x^2$ for the same limits:

$$V_x = \pi \int_1^2 (x^2)^2 dx = \pi \left[\frac{x^5}{5} \right]_1^2 = \pi \left[\frac{(2)^5}{5} - \frac{(1)^5}{5} \right] = \frac{31\pi}{5}$$

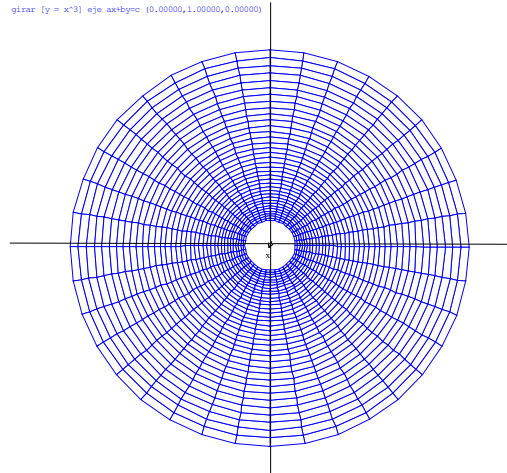
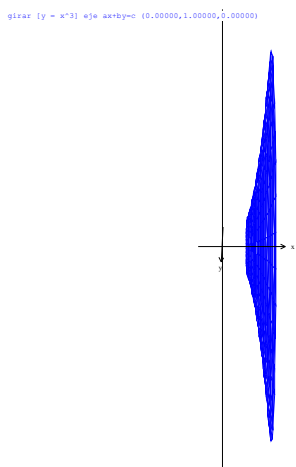


Now in this graph we can observe how it goes from $x = 1$ to $x = 2$.

Now I will continue with $y = x^3$.

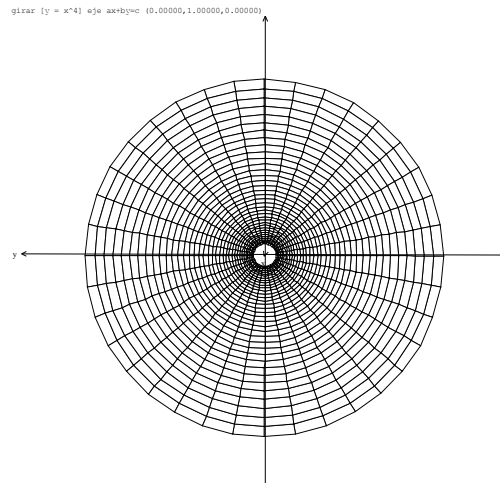
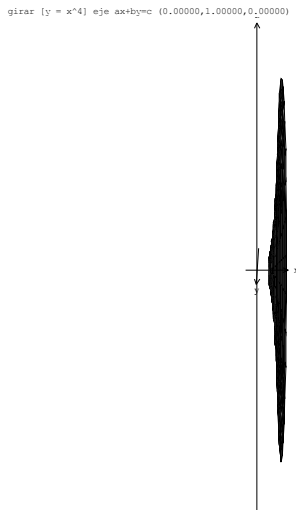
Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

$$V_x = \pi \int_1^2 (x^3)^2 dx = \pi \left[\frac{x^7}{7} \right]_1^2 = \pi \left[\frac{(2)^7}{7} - \frac{(1)^7}{7} \right] = \frac{127\pi}{7}$$



$$y = x^4$$

$$V_x = \pi \int_1^2 (x^4)^2 dx = \pi \left[\frac{x^9}{9} \right]_1^2 = \pi \left[\frac{(2)^9}{9} - \frac{(1)^9}{9} \right] = \frac{511\pi}{9}$$



From the graphs it can be concluded that as the function elevates its power, the volume closes more.

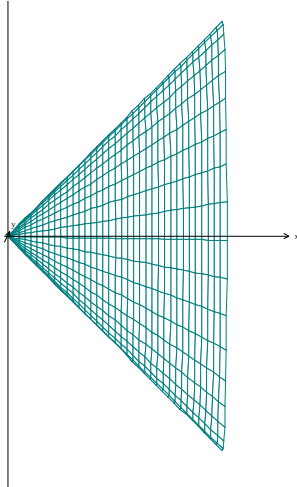
Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

Now the limits will be from $a = 0$ to $b = 3$.

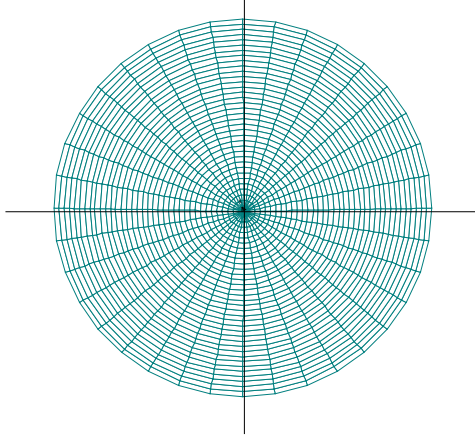
$$y = x$$

$$V_x = \pi \int_0^3 (x)^2 dx = \pi \left[\frac{x^3}{3} \right]_0^3 = \pi \left[\frac{(3)^3}{3} - \frac{(0)^3}{3} \right] = \frac{27\pi}{3}$$

gizar [y = x] eje ax+by=c (0.00000,1.00000,0.00000)

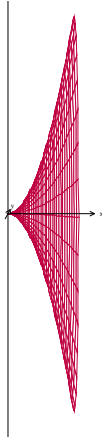


gizar [y = x] eje ax+by=c (0.00000,1.00000,0.00000)

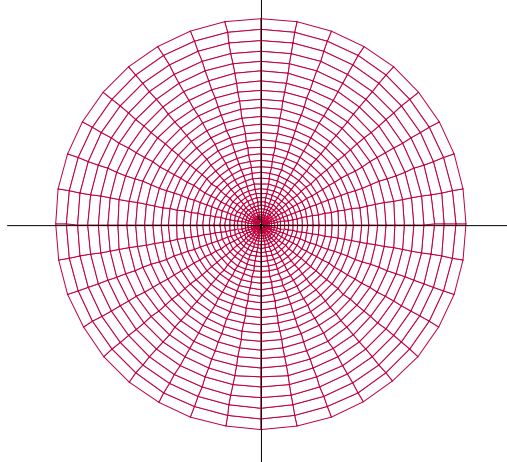


$$V_x = \pi \int_0^3 (x^2)^2 dx = \pi \left[\frac{x^5}{5} \right]_0^3 = \pi \left[\frac{(3)^5}{5} - \frac{(0)^5}{5} \right] = \frac{243\pi}{5}$$

gizar [y = x^2] eje ax+by=c (0.00000,1.00000,0.00000)



gizar [y = x^2] eje ax+by=c (0.00000,1.00000,0.00000)



Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

Therefore, a general conjecture for the volumes of revolution of x would be:

$$V_x = \frac{(b^{2n+1} - a^{2n+1})\pi}{2n+1}$$

Let's prove the conjecture with a random example:

$$y = x^3, a = 2, b = 5$$

So, by the use of the formula the result would be this:

$$V_x = \pi \int_2^5 (x^3)^2 dx = \pi \left[\frac{x^7}{7} \right]_2^5 = \pi \left[\frac{(5)^7}{7} - \frac{(2)^7}{7} \right] = \frac{77997\pi}{7}$$

And by the using of the conjecture, the result gives this:

$$V_x = \frac{((5)^{2(3)+1} - (2)^{2(3)+1})\pi}{2(3)+1} = \frac{77997\pi}{7}$$

Therefore, the conjecture for volumes of revolution generated by region B to the x-axis is correct.

Now, is there a general formula for ratios of volumes of revolution generated by region A to the y-axis?

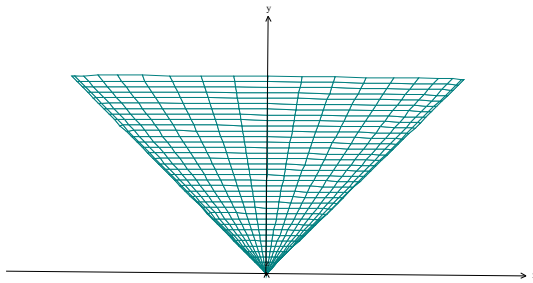
Foe the development of this conjecture, I will use the same examples used before to find a conjecture for the x-axis.

$$y = x, a = 0, b = 1$$

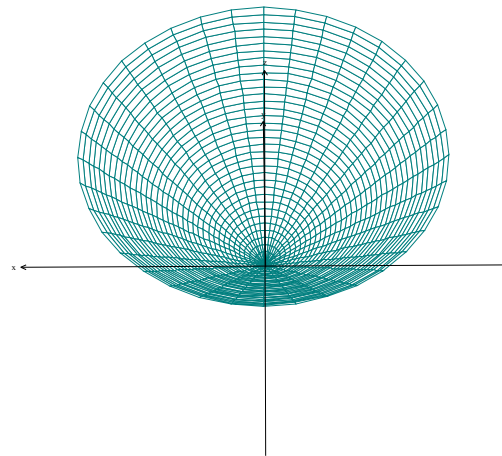
$$V_y = \pi \int_0^1 (y)^2 dy = \pi \left[\frac{y^3}{3} \right]_0^1 = \pi \left[\frac{(1)^3}{3} - \frac{(0)^3}{3} \right] = \frac{\pi}{3}$$

Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

gizar [y = x] eje ax+by=c (1.00000,0.00000,0.00000)



gizar [y = x] eje ax+by=c (1.00000,0.00000,0.00000)

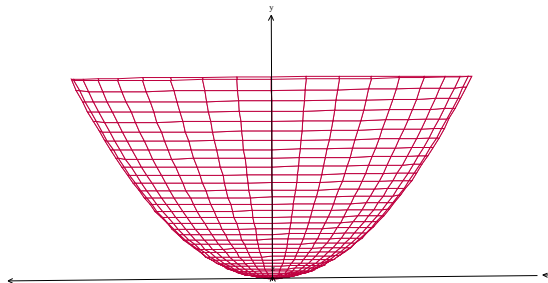


So, as we can see the area now rotates around the y-axis, as the region A is the one that it is spinning now.

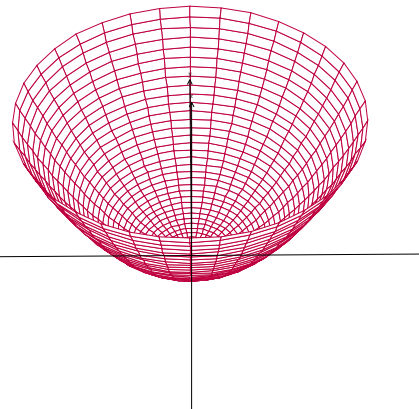
$$y = x^2$$

$$V_y = \pi \int_0^1 \left(y^{\frac{1}{2}}\right)^2 dy = \pi \left[\frac{y^2}{2}\right]_0^1 = \pi \left[\frac{(1)^2}{2} - \frac{(0)^2}{2}\right] = \frac{\pi}{2}$$

gizar [y = x^2] eje ax+by=c (1.00000,0.00000,0.00000)



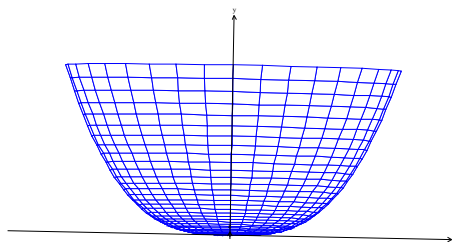
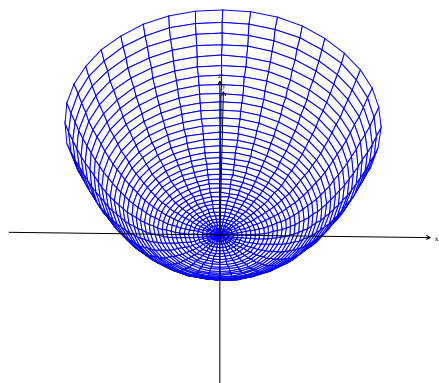
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Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

$$y = x^3$$

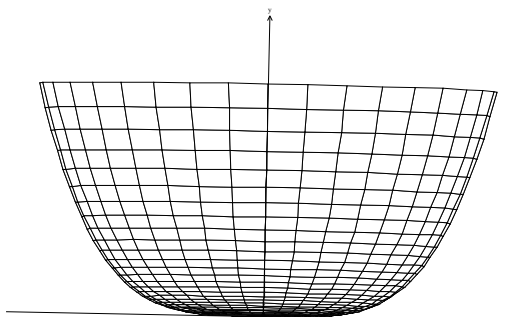
$$V_y = \pi \int_0^1 \left(y^{\frac{1}{3}}\right)^2 dy = \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^1 = \pi \left[\frac{3}{5} (1)^{\frac{5}{3}} - \frac{3}{5} (0)^{\frac{5}{3}}\right] = \frac{3\pi}{5}$$



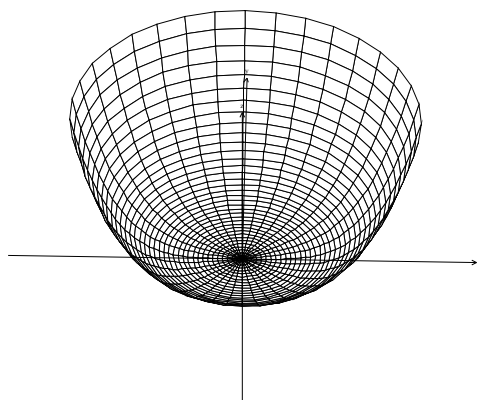
As it may be observed, as the power of the function is increased, the volume gets wider. This is opposed to the volume of revolution formed to the x-axis, which gets narrower when the exponent increases. Those two situations are connected, as when the volume of y gets wider, limits the space of the other volume and vice versa.

$$V_y = \pi \int_0^1 \left(y^{\frac{1}{4}}\right)^2 dy = \pi \left[\frac{2}{3} y^{\frac{3}{2}}\right]_0^1 = \pi \left[\frac{2}{3} (1)^{\frac{3}{2}} - \frac{2}{3} (0)^{\frac{3}{2}}\right] = \frac{2\pi}{3}$$

gizar [y = x^4] eje axby=c (1.00000,0.00000,0.00000)



gizar [y = x^4] eje axby=c (1.00000,0.00000,0.00000)



Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

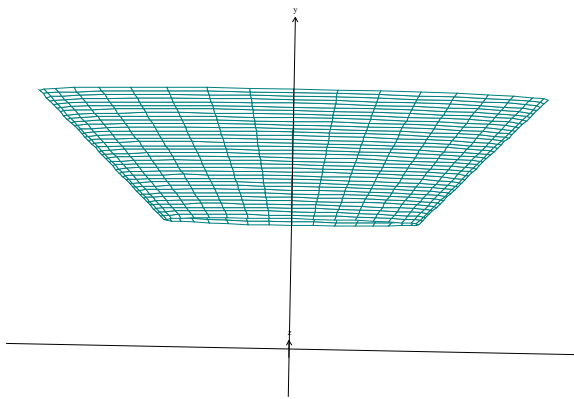
Now, the limits of the examples will be changed.

$$a = 1, b = 2$$

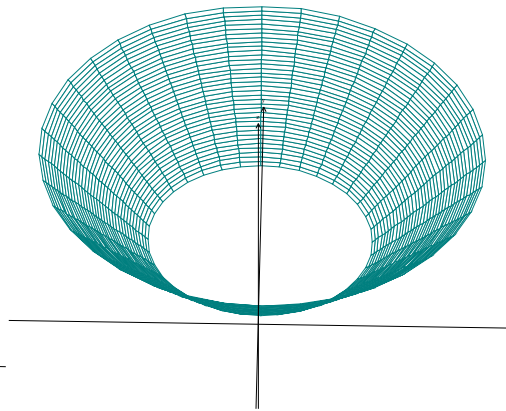
$$y = x$$

$$V_y = \pi \int_1^2 (y)^2 dy = \pi \left[\frac{y^3}{3} \right]_1^2 = \pi \left[\frac{(2)^3}{3} - \frac{(1)^3}{3} \right] = \frac{7\pi}{3}$$

gizar [y = x] eje axrby= (1.00000,0.00000,0.00000)



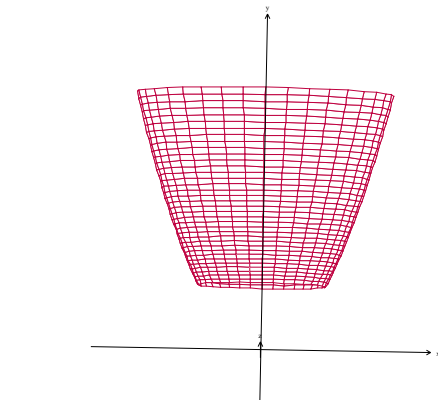
gizar [y = x] eje axrby= (1.00000,0.00000,0.00000)



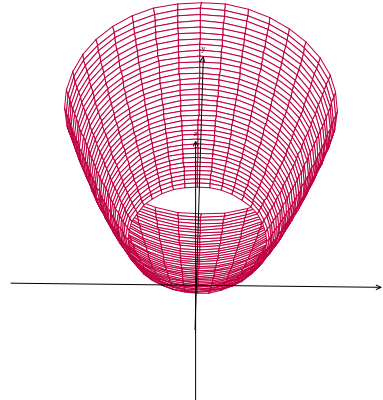
$$y = x^2$$

$$V_y = \pi \int_1^2 \left(y^{\frac{1}{2}} \right)^2 dy = \pi \left[\frac{y^2}{2} \right]_1^2 = \pi \left[\frac{(2)^2}{2} - \frac{(1)^2}{2} \right] = \frac{3\pi}{2}$$

gizar [y = x^2] eje axrby= (1.00000,0.00000,0.00000)



gizar [y = x^2] eje axrby= (1.00000,0.00000,0.00000)



Baca Gonzalez, Gabriel Alejandro2106-005 May 2009

$$y = x^3$$

$$V_y = \pi \int_1^2 \left(y^{\frac{1}{3}}\right)^2 dy = \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_1^2 = \pi \left[\frac{3}{5} (2)^{\frac{5}{3}} - \frac{3}{5} (1)^{\frac{5}{3}}\right] = \frac{3\pi}{5}$$