

Type 1: Investigating a sequence of Numbers

This is an investigation about series and sequences involving permutations. From a given series, I find the pattern of numbers that result from different values and use graphs to conjecture an expression from the series. By using mathematical induction and direct proof, I prove the general terms that I derived for the series.

Part 1:

The sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is defined by

$$a_1 = 1 \times 1!, a_2 = 2 \times 2!, a_3 = 3 \times 3!, \dots$$

\therefore From the pattern of different values of n in a_n above, I conclude that $a_n = n \times n!$

Part 2:

Let $S_n = a_1 + \dots + a_n$

If $n=1$

$$S_1 = a_1 \text{ where } a_1 = 1 \times 1!$$

$$S_1 = 1 \times 1!$$

$$S_1 = 1$$

If $n=2$

$$S_2 = a_1 + a_2 \text{ where } a_1 = 1 \times 1!, a_2 = 2 \times 2!$$

$$S_2 = 1 \times 1! + 2 \times 2!$$

$$S_2 = 1 + (2 \times (2 \times 1))$$

$$S_2 = 1 + 4$$

$$S_2 = 5$$

If $n=3$

$$S_3 = a_1 + a_2 + a_3 \text{ where } a_1 = 1 \times 1!, a_2 = 2 \times 2!, a_3 = 3 \times 3!$$

$$S_3 = 1 \times 1! + 2 \times 2! + 3 \times 3!$$

$$S_3 = 1 + (2 \times (2 \times 1)) + (3 \times (3 \times 2 \times 1))$$

$$S_3 = 1 + 4 + 18$$

$$S_3 = 23$$

Part 3:

From Part 2, I know that:

$$S_n = 1 + 4 + 8 + \dots + n \times n!$$

To conjecture an expression of S_n , I first organize the results that are derived in Part 2 to discover a pattern in the value of S_n as n increases.

Table 1.1:

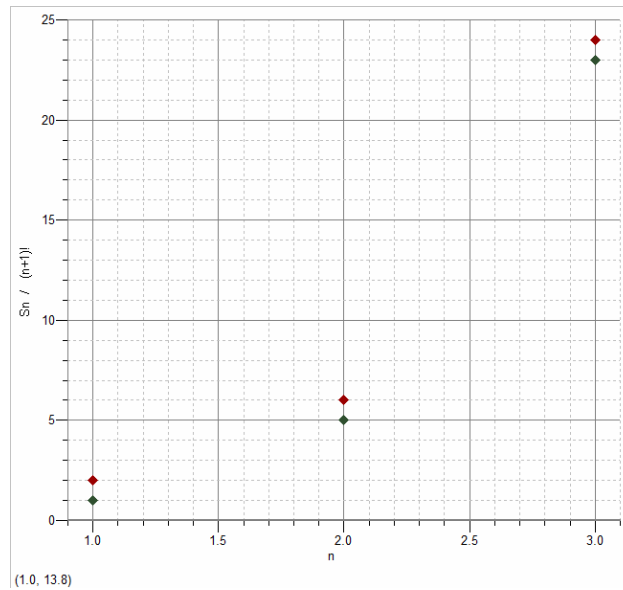
n	a_n	S_n
1	1	1
2	4	5
3	18	23
-	-	-
-	-	-
n	$n \times n!$?

The same results of S_n from Table 1.1 can be represented as follows:

Table 1.2 :

n	S_n
1	$2-1 = (1+1)! - 1 = 1$
2	$6-1 = (2+1)! - 1 = 5$
3	$24-1 = (3+1)! - 1 = 23$
-	-
-	-
n	$(n+1)! - 1$

\therefore From the patterns exhibited in Table 1.2, I notice that $S_n = (n+1)! - 1$ which is further illustrated in *Graph 1.1*.



Graph 1.1

In *Graph 1.1*, I plotted the graph of S_n for the first three values (represented by green dots) and I assumed that $(n+1)!$ will lead to a conjecture for S_n and plotted its values for $n=1,2,3$ (represented by red dots). From the two graphs, I notice that $(n+1)!$ is exactly 1 unit above S_n for all three points \therefore I conclude that: $S_n = (n+1)! - 1$

Part 4:

The conjecture that I derived in Part 3 for S_n can be proven through Mathematical Induction:

$$S_n \text{ is } "1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1"$$

(1) If $n=1$

$$\text{LHS: } 1 \times 1! = 1$$

$$\text{RHS: } (1+1)! - 1 = 1$$

$$\text{RHS} = \text{LHS} = S_1 = 1$$

$\therefore S_1$ is true

(2) If S_k is true, then

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k+1)! - 1$$

If $k=k+1$, then

$$S_{k+1} = ((k+1)+1)! - 1$$

$$S_{k+1} = (k+2)! - 1$$

$$\text{Now, } S_{k+1} = 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1)(k+1)!$$

$$= S_k + a_{k+1}$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)!((k+1)+1) - 1$$

$$= (k+2)! - 1 \quad \square \quad (k+1)!((k+1)+1) = (k+2)!$$

Thus S_{k+1} is true whenever S_k is true and S_1 is true.

$\therefore S_n$ is true for all n

Part 5:

\square $a_n = n \times n!$, I use this formula to show that $a_n = (n+1)! \times n!$ is also true by simplifying $(n+1)! \times n!$ to equal $n \times n!$

$$\begin{aligned} n \times n! &= (n+1)! - n! \\ &= (n+1)n! - n! \quad \square \quad (n+1)! = (n+1)n! \\ &= n! (n+1-1) \\ &= n \times n! \end{aligned}$$

$$\therefore a_n = (n+1)! - n!$$

Now, I use $a_n = (n+1)! \times n!$ to devise a direct proof for the expression of $S_n = (n+1)! - 1$ that I conjectured in Part 3.

$$\begin{aligned} \square \quad a_n &= (n+1)! - n! \\ a_1 &= (1+1)! - 1! = 2! - 1! \\ a_2 &= (2+1)! - 2! = 3! - 2! \\ a_3 &= (3+1)! - 3! = 4! - 3! \end{aligned}$$

From Part 3, I know that $S_n = a_1 + a_2 + a_3 \dots + a_n$

$$\therefore S_n = 2! - 1! + 3! - 2! + 4! - 3! \dots + n! - (n-1)! + (n+1)! - n!$$

When the first value of the first term is subtracted from the second value of the following term, 0 is derived so I cancel these terms. After I cancel the values to the most simplified manner, $-1!$ and $(n+1)!$ are left in the expression from which the following equation is derived:

$$S_n = (n+1)! - 1$$

\therefore The conjecture of $S_n = (n+1)! - 1$ is proven

Part 6:

From Part 5, I know that:

$$a_n = (n+1)! - n!$$

I use the equation of a_n to derive an expression for a_{n+1} by substituting $n+1$ for n and simplify it:

If $n=n+1$, then

$$a_{n+1} = ((n+1)+1)! - (n+1)!$$

$$= (n+2)! - (n+1)!$$

$$\text{Let } c_n = a_n + a_{n+1}$$

To express c_n in factorial notation, I substitute a_n and a_{n+1} with their equivalent factorial notation forms and simplify them:

$$c_n = (n+1)! - n! + (n+2)! - (n+1)!$$

$$c_n = (n+2)! - n!$$

$$\therefore C_n = (n+2)! - n!$$

Part 7:

$$\text{Let } T_n = c_1 + c_2 + \dots + c_n$$

If $n=1$

$$T_1 = c_1 \text{ where } c_1 = (1+2)! - 1!$$

$$T_1 = (1+2)! - 1!$$

$$T_1 = (3 \times 2 \times 1) - 1$$

$$T_1 = 6 - 1$$

$$T_1 = 5$$

If $n=2$

$$T_2 = c_1 + c_2 \text{ where } c_1 = (1+2)! - 1!, c_2 = (2+2)! - 2!$$

$$T_2 = (1+2)! - 1! + (2+2)! - 2!$$

$$T_2 = (3 \times 2 \times 1) - 1 + (4 \times 3 \times 2 \times 1) - (2 \times 1)$$

$$T_2 = (6 - 1) + (24 - 2)$$

$$T_2 = 5 + 22$$

$$T_2 = 27$$

If $n=3$

$$T_3 = c_1 + c_2 + c_3 \text{ where } c_1 = (1+2)! - 1!, c_2 = (2+2)! - 2!, c_3 = (3+2)! - 3!$$

$$T_3 = (1+2)! - 1! + (2+2)! - 2! + (3+2)! - 3!$$

$$T_3 = (3 \times 2 \times 1) - 1 + (4 \times 3 \times 2 \times 1) - (2 \times 1) + (5 \times 4 \times 3 \times 2 \times 1) - (3 \times 2 \times 1)$$

$$T_3 = (6 - 1) + (24 - 2) + (120 - 6)$$

$$T_3 = 5 + 22 + 114$$

$$T_3 = 141$$

Part 8:

From Part 7:

$$T_n = c_1 + c_2 + \dots + c_n$$

$$T_n = 5 + 22 + 114 + \dots + (n+2)! - n!$$

To conjecture an expression of T_n , I first organize the results that are derived in Part 7 to discover a pattern in the value of T_n as n varies.

Table 2.1:

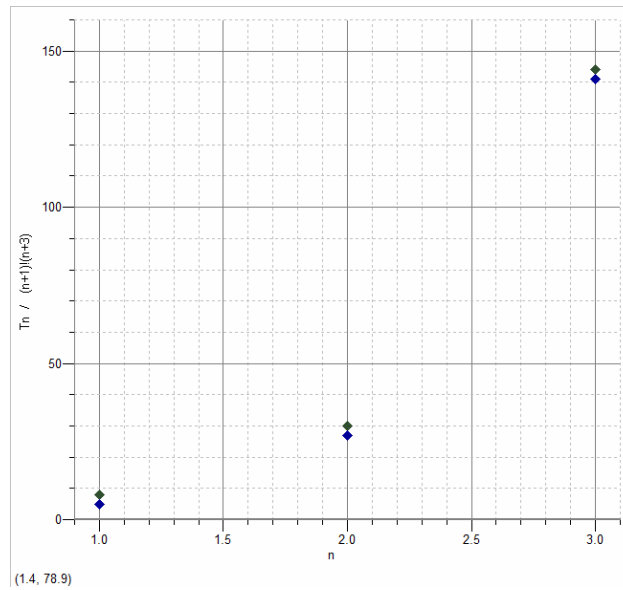
n	C_n	T_n
1	5	5
2	22	27
3	114	141
-	-	-
-	-	-
n	$(n+2)! - n!$?

The same results of T_n from Table 2.1 can be represented as follows:

Table 2.2:

n	T_n
1	$2 \times 4 - 3 = 5 = (1+1)!(1+3) - 3$
2	$6 \times 5 - 3 = 27 = (2+1)!(2+3) - 3$
3	$24 \times 6 - 3 = 141 = (3+1)!(3+3) - 3$
-	-
-	-
n	$(n+1)!(n+3) - 3$

Thus, from the pattern exhibited in Table 2.2, I notice that $T_n = (n+1)!(n+3) - 3$ which is further illustrated by Graph 2.1.



Graph 2.1

In Graph 2.1, I plotted the graph of T_n for the first three values (represented by blue dots) and I assumed that $(n+1)!(n+3)$ will lead to a conjecture for formula for S_n and plotted its values for $n=1,2,3$ (represented by green dots). From the two graphs, I notice that $(n+1)!(n+3)$ is exactly greater by 3 units to T_n for all three points \therefore I conclude that: $S_n = (n+1)!(n+3) - 3$

Part 9:

The conjecture that I derived in Part 7 for T_n can be proven through Mathematical Induction:

$$T_n \text{ is: } c_1 + c_2 + \dots + c_n = (n+1)!(n+3) - 3$$

$$T_n = (1+2)!-1! + (2+2)!-2! + \dots + (n+2)!-n! = (n+1)!(n+3) - 3$$

(1) If $n=1$

$$\text{LHS: } c_1$$

$$= (1+2)!-1!$$

$$= 5$$

$$\text{RHS: } (1+1)!(1+3) - 3 = 5$$

$$\text{LHS} = \text{RHS} = T_1 = 5$$

$\therefore T_1$ is true

(2) If T_k is true, then

$$c_1 + c_2 + \dots + c_k = (k+1)!(k+3) - 3$$

$$(1+2)!-1! + (2+2)!-2! + \dots + (k+2)!-k! = (k+1)!(k+3) - 3$$

If $k=k+1$, then

$$S_{k+1} = ((k+1)+1)!((k+1)+3) - 3$$

$$S_{k+1} = (k+2)!(k+4) - 3$$

$$\text{Now, } S_{k+1} = S_k + c_{k+1}$$

$$\begin{aligned}
 &= (k+1)! (k+3) - 3 + (k+3)! - (k+1)! \\
 &= (k+1)! ((k+3) - 1) - 3 + (k+3)! \\
 &= (k+1)!(k+2) - 3 + (k+3)! \\
 &= (k+2)!(k+4) - 3 \quad (k+1)!(k+2) + (k+3)! = (k+2)!(k+4)
 \end{aligned}$$

Thus T_{k+1} is true whenever T_k is true and T_1 is true.

$\therefore T_n$ is true for all n

Conclusion:

Through this investigation, I have developed my knowledge about series and sequences involving permutations. I have learnt to use the patterns in a series to conjecture an expression for it and I had an opportunity to utilize my awareness of mathematical induction into proving the general term for the series. Most importantly, I have learnt to use technology related to series involving permutations. I enjoyed this investigation.