Mathematics Folio: Infinite Surds Michael Zhao
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The aim of this folio is to explore the nature of infinite surds. Infinite surds take the form $f(x) = \sqrt{k + \sqrt{k + \sqrt{k \ldots}}}$.

Part one: $\sqrt{1 + \sqrt{1}}$

Consider the surd where:

$$a_1 = \sqrt{1 + \sqrt{1}}$$

$$a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$$

$$a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}$$

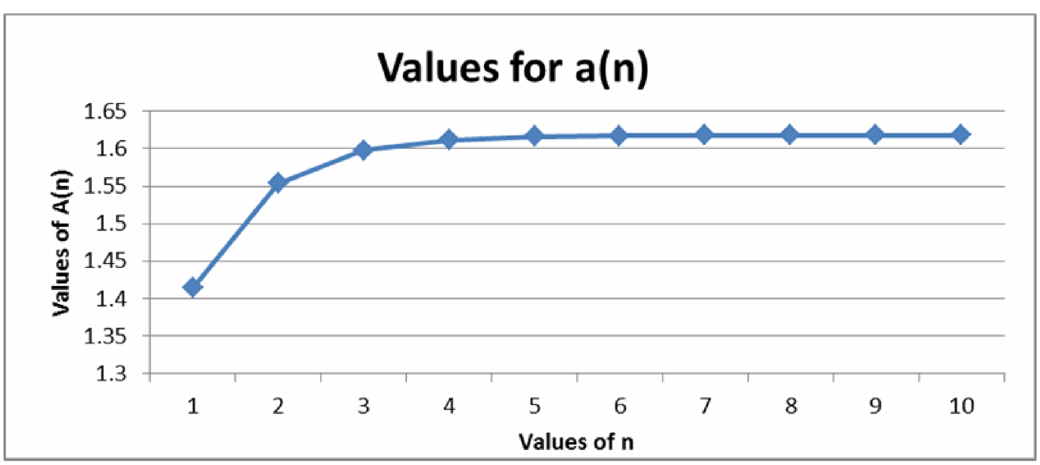
Etc.

To begin, a recursive rule was determined from observing what changed from a_1 to a_2 and so on. The major change was that the previous surd was added to 1 and then square rooted to provide the next value. This observation provided the basis for the recursive rule below.

$$a_{n+1} = \sqrt{1 + a_n}$$

The decimal values were calculated for a_1 to a_{10} and the results were then graphed.

n	a(n)	a(n) - a(n-1)
1	1.414214	-0.139560412
2	1.553774	-0.044279208
3	1.598053	-0.013794572
4	1.611848	-0.004273452
5	1.616121	-0.001321592
6	1.617443	-0.000408492
7	1.617851	-0.00012624
8	1.617978	-3.90113E-05
9	1.618017	-1.20552E-05
10	1.618029	-3.72528E-06



It can be observed that the slope of the function gradually decreases asymptotes towards a value of y. The values of $a_n - a_{n+1}$ suggest that as n becomes larger, the function asymptotes towards a certain value. Finding the exact value of the surd requires being able to solve the surd. Solving the surd was done in the following method.

$$Let\ x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1\ldots}}}}$$

$$\therefore x^2 = 1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1\ldots}}}}$$

$$\therefore x^2 = 1 + x$$

$$\therefore -x^2 + x + 1 = 0$$

$$\therefore x^2 - x - 1 = 0$$

$$\therefore x^2 - x + \frac{1}{4} - \frac{1}{4} - 1 = 0$$

$$\therefore \left(x - \frac{1}{2}\right)^2 - \frac{5}{4} = 0$$

$$\therefore \left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\therefore x - \frac{1}{2} = \pm\sqrt{\frac{5}{4}}$$

$$\therefore x = \frac{1}{2} \pm \sqrt{\frac{5}{4}}$$

$\therefore$ The exact value of the infinite surd $a_n = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1\ldots}}}}$ is $\frac{1}{2} + \sqrt{\frac{5}{4}}$.

Only the positive is used as you cannot have a real negative surd. In addition, $\frac{1}{2} + \sqrt{\frac{5}{4}} \approx$ 1.61803398 which is very close to the value for $n = 10$ however the gaps between the values for n become smaller and the difference between $n = 10$ and $n = 11$ is approximately -3.72528E-06 which is a minute change compared to the gap between $n = 10$ and the value at which $n = \infty$. These results support the answer which was provided alge braically due to the small margin between the calculated and algebraic results.

Part Two: $\sqrt{2+\sqrt{2}}$

Consider another infinite surd:

$$a_n = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2\ldots}}}}$$

Where the first term is:

$$a_1 = \sqrt{2+\sqrt{2}}$$

Repeat the entire process above to find the exact value for this surd.

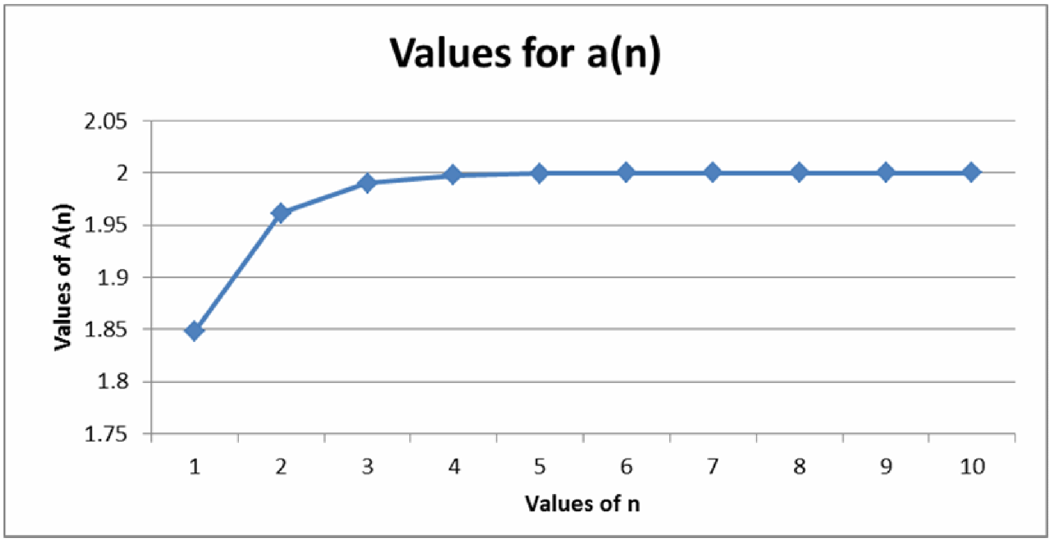
A recursive rule is devised from the observations made between terms in the series.

The major observation which identified the difference and determined the change between values of n was that the original surd is added to a 2 and then square rooted much like the first infinite surd which was provided.

$$a_{n+1} = \sqrt{2+a_n}$$

A table of values is composed for values of n in addition to a graph to identify possible asymptotes.

n	a(n)	a(n) - a(n-1)
1	1.847759065	-0.113811496
2	1.961570561	-0.028798893
3	1.990369453	-0.007221459
4	1.997590912	-0.001806725
5	1.999397637	-0.000451766
6	1.999849404	-0.000112947
7	1.999962351	-2.82371E-05
8	1.999990588	-7.05928E-06
9	1.999997647	-1.76482E-06
10	1.999999412	-4.41206E-07



It can be observed that the values seem to approach 2 as n becomes larger. The graph has a very similar shape to the graph in Part One but with a sharper turn and the values for $a_n - a_{n+1}$ also have a similar pattern in that they gradually decrease and become very small as n becomes larger. However, the values are different in that they become more distinct. E.g. the big gaps are bigger, smaller gaps are smaller. To solve, a similar method to the method used in Part one will be used.

$$Let\ x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2\ldots}}}}$$

$$\therefore x^2 = 2+\sqrt{2+\sqrt{2+\sqrt{2\ldots}}}$$

$$\therefore x^2 = 2 + x$$

$$\therefore x^2 - x - 2 = 0$$

$$\therefore x^2 + x + \frac{1}{4} - \frac{1}{4} + 2 = 0$$

$$\therefore \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = 0$$

$$\therefore \left(x - \frac{1}{2}\right)^2 = \frac{9}{4}$$

$$\therefore x - \frac{1}{2} = \pm\sqrt{\frac{9}{4}}$$

$$\therefore x = \frac{1}{2} \pm \sqrt{\frac{9}{4}}$$

$$\therefore x = \frac{1}{2} \pm \frac{3}{2}$$
$$\therefore x = \frac{1}{2} + \frac{3}{2}$$
$$\therefore x = 2$$

The exact value for the infi nite surd is 2 where $n = \infty$. These results correspond to the calculated results validating the observation that the infinite surd approached 2 with greater values of n.

Part Three: The General Infinite Surd - $\sqrt{k+\sqrt{k}}$

Consider the General Infinite Surd $\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k\ldots}}}}$ *where the first term is* $\sqrt{k+\sqrt{k}}$.

Find an expression for the exact value of this general infinite surd in terms of k.

$$Let\ x = \sqrt{k+\sqrt{k+\sqrt{k\ldots}}}$$
$$\therefore x^2 = k + x$$
$$\therefore x^2 - x - k = 0$$
$$\therefore \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - k = 0$$
$$\therefore \left(x - \frac{1}{2}\right)^2 = k + \frac{1}{4}$$
$$\therefore x - \frac{1}{2} = \sqrt{k + \frac{1}{4}}$$
$$\therefore x = \frac{1}{2} \pm \sqrt{k + \frac{1}{4}}$$

However, only the positive is used as negative surds are not real and canno t be put in the place of k.

$$\therefore x = \frac{1}{2} + \sqrt{k + \frac{1}{4}}$$

The exact value for the General Infinite Surd can be expressed as $x = \frac{1}{2} + \sqrt{k + \frac{1}{4}}$ where x represents the value of the infinite surd.

The value of an infinite surd is not always an integer.

Find some values of k that make the expression an integer. Find the general statement that represents all values for k is for which the expression is an integer.

The first value of k which is an integer is 2 which is equal to 2 when $n = \infty$ due to the results of Part Two. The next integer value would be equal to 3. It can be worked out using the value for the General Infinite Surd.

Find k when x is equal to 3.

$$3 = \frac{1}{2} + \sqrt{k + \frac{1}{4}}$$

$$2.5 = \sqrt{k + \frac{1}{4}}$$

$$6.25 = k + \frac{1}{4}$$

$$k = 6.$$

The value for k where the value of the infinite surd is equal to 1 can also be worked out.

$$1 = \frac{1}{2} + \sqrt{k + \frac{1}{4}}$$

$$0.5 = \sqrt{k + \frac{1}{4}}$$

$$0.25 = k + \frac{1}{4}$$

$$k = 0$$

Finally, the value for k where the value of the infinite surd is equal to 4 can be worked out in order to see if there is a clear pattern.

$$4 = \frac{1}{2} + \sqrt{k + \frac{1}{4}}$$

$$3.5 = \sqrt{k + \frac{1}{4}}$$

$$12.25 = k + \frac{1}{4}$$

$$k = 12$$

The results are tabulate d to see if there is a correlation between the value for k and the integer value of the surd.

Integer Value (x)	k
1	0
2	2
3	6
4	12
5	20
6	30
7	42
8	56
9	72
10	90

The table shows how there is a growth in the value of k which correlates directly to the integer values of the infinite surd. The correlation can be represented by the formula $k = (x-1)^2 + (x-1)$ or $x = \frac{1-\sqrt{4k+1}}{2}$ or with the General expression for Infinite surds .

When put into an excel spread sheet, and teste d, the values for x all returned with the matching values for k. Those values of k when substituted into the table of values for a_n showed values which all approached integers which were the original x values. Examples of these results are shown below.

$$k = 420$$

$$x = \frac{1-\sqrt{4k+1}}{2}$$

$$x = 21$$

n	a(n)
1	20.9879465773076
2	20.9997130117844
3	20.9999931669461
4	20.9999998373082

5	20.9999999961264
6	20.9999999999078
7	20.9999999999978

$k = 182$

$$x = \frac{1 - \sqrt{4k+1}}{2}$$

$x = 14$

n	a(n)
1	13.9818002261237
2	13.9993499929862
3	13.9999767854445
4	13.9999991709087
5	13.9999999703896
6	13.9999999989425
7	13.9999999999622

$k = 650$

$$x = \frac{1 - \sqrt{4k+1}}{2}$$

$x = 26$

n	a(n)
1	25.99028852413850
2	25.99981324017810
3	25.99999640846470
4	25.99999993093200
5	25.99999999867180
6	25.99999999997450
7	25.99999999999950

Infinite Surds and their traits are all closely interconnected with each other. The value of the infinite surd and the value for k in the infinite surd have a unique correlation which means that they can easily be calculated at any time as long as one variable is at hand. In addition, the general expression of an Infinite Surd can be used to calculate the value of any surd effortlessly.