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MATH PORTFOLIO
INFINTE SURDS

$$a_n = \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \ldots\ldots\infty}}}}$$

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Math 20 IB

Presented To Ms. Garrett

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Surds are used commonly in math, they just are not referred to as surds. A surd is any positive number that is in square root form. Once you simplify the surd it must form a positive irrational number. If a rational number is formed, it is not considered to be a surd.

Example:

a) $\sqrt{3}$ = 1.73205... is a surd

b) $\sqrt{9}$ = 3 is not a surd

c) $\sqrt{5}$ = 2.23606... is a surd

Infinite surds are just surds forming a sequence that goes on forever. The exact value of an infinite surd is expressed in the square root form. When the infinite surds in those sequences are simplified, they are allowed to be rational or irrational unlike a surd.

The following is an the first example of an infinite surd:

Example 1:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

You may be wondering why this can be classified as a surd when $\sqrt{1}$ is not a surd. $\sqrt{1}$ simplified forms a natural number and cannot be classified as a surd. As you can see in the first 10 terms of the infinite surd, they are all irrational numbers.

a_1: $\sqrt{\underbrace{1+\sqrt{1}}_{\text{Given}}}$ = 1.414213 ...

a_2: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1}}}_{a_1}}$ = 1.553773 ...

a_3: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1}}}}_{a_2}}$ = 1.598053 ...

a_4: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}_{a_3}}$ = 1.611847 ...

a_5: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}_{a_4}}$ = 1.616121 ...

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a_6: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}_{a_5}}$ = 1.617442 ...

a_7: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}_{a_6}}$ = 1.617851 ...

a_8: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}_{a_7}}$ = 1.617977 ...

a_9: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}}_{a_8}}$ = 1.618016 ...

a_{10}: $\sqrt{1+\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}}}_{a_9}}$ = 1.618028 ...

From the first ten terms of the sequence you can see that the next sequence is $\sqrt{1+\text{the previous term}}$. Turning that into a formula for a_{n+1} in terms of a_n makes:

Formula 1a:

$a_{n+1}=\sqrt{1+a_n}$

When looking for the next term in the sequence of this infinite surd, you place the term number that you want to find subtracted it by 1 for n.

Example 1a:

$a_1=\sqrt{1+\sqrt{1}}$

Find a_2 using the formula:

$a_{n+1}=\sqrt{1+a_n}$

$a_{(2-1)+1}=\sqrt{1+a_{(2-1)}}$

$a_{(1)+1}=\sqrt{1+a_1}$

$a_2=\sqrt{1+\underbrace{\sqrt{1+\sqrt{1}}}_{a_1}}$

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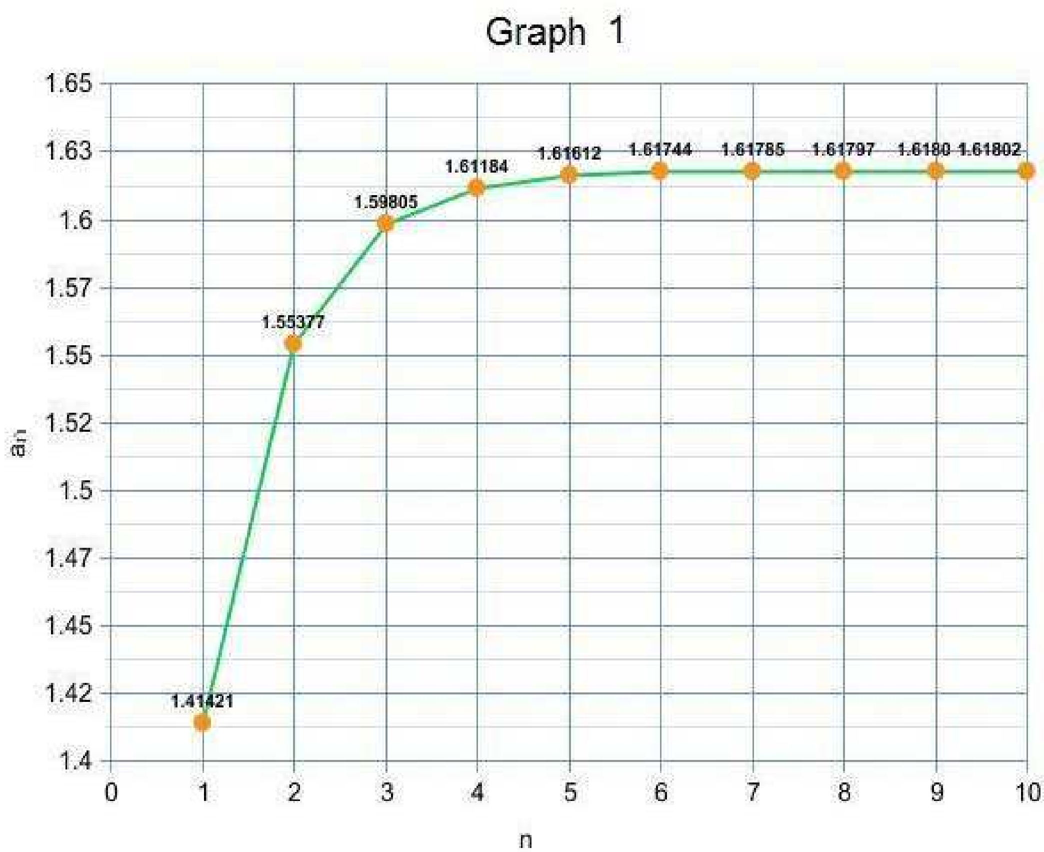

From the plotted points of the infinite surd $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}}$ in graph 1, you can see that the greater n increases, the closer a_n gets to the value of about 1.61803. But a_n never touches that value; that causes the curve to flatten out. It also shows that the rise of the slope is continually decreasing as n increase. So as n gets larger, $a_n - a_{n+1}$ gets closer to 0. Since the sequence goes on forever, it cannot be determined if $a_n - a_{n+1}$ ever equals 0.

Discovery 1a:
as $n \to \infty$, $a_n - a_{n+1} \to 0$

Example 1b:
$a_4 - a_5$

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}} - \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}$$

1.6118 - 1.6161
= -0.0043

$a_6 - a_7$

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}} - \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}$$

1.6174 - 1.6179
= -0.0005

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From previous results, you know that as $n \to \infty$, a_n gets flatter and levels out right under about 1.61803. The value of the infinite surd which is about 1.61803 can be considered as x. So, let x be:

$$x = \sqrt{1+\sqrt{1+\dots}}$$

$$(x)^2 = \left(\sqrt{1+\sqrt{1+\dots}}\right)^2$$ Square both sides to create an equation to work with.

$$x^2 = 1+\sqrt{1+\sqrt{1+\dots}}$$ The infinite surd continues.

$$x^2 = 1+x$$ Substitute $\sqrt{1+\sqrt{1+\dots}}$ with x since $x = \sqrt{1+\sqrt{1+\dots}}$.

$$x^2 - x - 1 = 0$$ Equate to zero.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Use quadratic formula to solve for x.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1+4}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$ Since a negative number cannot be square rooted, the negative exact value is an extraneous root because it doesn't work.

$$x = \frac{1 + \sqrt{5}}{2}$$ Only the positive value is accepted.

Discovery 1b:

$$x = \sqrt{1+\sqrt{1+\dots}} = \frac{1+\sqrt{5}}{2}$$

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We can further our understanding of infinite surds by looking at this next example:

Example 2:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}}$$

Like the first infinite surd, the first 10 terms of this infinite surd's sequence are also all irrational numbers.

a_1: $\underbrace{\sqrt{2+\sqrt{2}}}_{\text{Given}}$ $= 1.847759\ \ldots$

a_2: $\sqrt{2+\underbrace{\sqrt{2+\sqrt{2}}}_{a_1}}$ $= 1.961570\ \ldots$

a_3: $\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2}}}}_{a_2}}$ $= 1.990369\ \ldots$

a_4: $\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}_{a_3}}$ $= 1.997590\ \ldots$

a_5: $\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}_{a_4}}$ $= 1.999397\ \ldots$

a_6: $\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}_{a_5}}$ $= 1.999849\ \ldots$

a_7: $\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}_{a_6}}$ $= 1.999984\ \ldots$

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$$a_8: \quad \sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}_{a_7}} \qquad = 1.999990 \ldots$$

$$a_9: \quad \sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}_{a_8}} \qquad = 1.999997 \ldots$$

$$a_{10}: \quad \sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}_{a_9}} \qquad = 1.999999 \ldots$$

From the first ten terms of the sequence you can see that the next sequence is $\sqrt{2+\text{the previous term}}$. Turning that into a formula for a_{n+1} in terms of a_n makes:

Formula 2a:

$a_{n+1} = \sqrt{2 + an}$

When looking for the next term in the sequence of this infinite surd, you place the term number that you want to find subtracted it by 1 for n.

Example 2a:

$a_1 = \sqrt{2+\sqrt{2}}$

Find a_4 using the formula:

$a_{n+1} = \sqrt{2 + an}$

$a_{(4-1)+1} = \sqrt{2 + a(4\ -1)}$

$a_{(3)+1} = \sqrt{2 + al}$

$$a_4 = \sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}_{a_3}}$$

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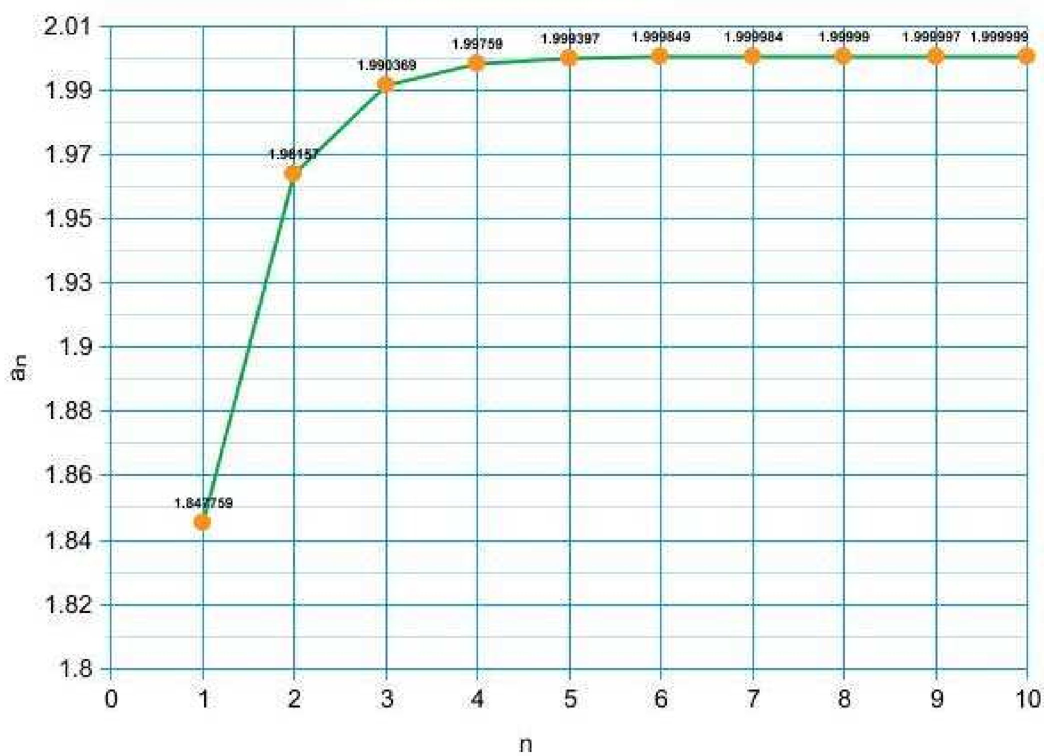
Graph 2



The relationship shown from the plotted points of the infinite surd $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}}$ in graph 2 is that as n increases, the closer a_n gets to the value of about 2.000 but a_n will never pass that point. The graph also shows how the rise of the slope is continually decreasing as n increase. So as n gets larger, $a_n - a_{n+1}$ continues to decrease closer to 0. Just like the infinite surd from example 1, it cannot be determined if $a_n - a_{n+1}$ ever equals 0 because the sequence also goes on forever.

Discovery 2a:
as $n \to \infty$, $a_n - a_{n+1} \to 0$

Example 2b:
$a_4 - a_5$

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} - \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$$

1.997590 - 1.999397
= -0.001807

$a_6 - a_7$

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} - \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}$$

1.999849 - 1.999984
= -0.000135

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From graph 2, you know that as $n \to \infty$, a_n gets flatter and levels out at about 2.0. The value of the infinite surd which is about 2.0 can be considered as *x*. So, let *x* be:

$$x = \sqrt{2+\sqrt{2+\ldots}}$$

$$(x)^2 = \left(\sqrt{2+\sqrt{2+\ldots}}\right)^2$$

Square both sides to create an equation to work with.

$$x^2 = 2+\sqrt{2+\sqrt{2+\ldots}}$$

The infinite surd continues.

$$x^2 = 2+x$$

Substitute $\sqrt{2+\sqrt{2+\ldots}}$ with *x* since $x = \sqrt{2+\sqrt{2+\ldots}}$.

$$x^2 - x - 2 = 0$$

Equate to zero.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use quadratic formula to solve for *x*.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1+8}}{2}$$

$$x = \frac{1 \pm \sqrt{9}}{2}$$

$$x = \frac{1 \pm 3}{2}$$

Since a negative number cannot be square rooted, the negative exact value is an extraneous root because it doesn't work.

$$x = \frac{4}{2}$$

$$x = 2$$

Only the positive value is accepted.

Discovery 2b:

$$x = \sqrt{2+\sqrt{2+\ldots}} = 2$$

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Now that two numerical examples of an infinite surd's exact value has been shown. Let us expand our understanding of the exact value of an infinite surd by looking at the following general infinite surd:

Example 3:

$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\ldots}}}}}$ and $a_1=\sqrt{k+\sqrt{k}}$

From graph 1 and graph 2 it is shown that an infinite surd's slope becomes flatter as the number of terms increases. Just like before, you can represent this whole infinite surd with:

$x=\sqrt{k+\sqrt{k+\ldots}}$

$(x)^2=\left(\sqrt{k+\sqrt{k+\ldots}}\right)^2$ Square both sides to create an equation to work with.

$x^2 = k+\sqrt{k+\sqrt{k+\ldots}}$ The infinite surd continues.

$x^2 = k+x$ Substitute $\sqrt{k+\sqrt{k+\ldots}}$ with x since $x=\sqrt{k+\sqrt{k+\ldots}}$.

$x^2 - x - k = 0$ Equate to zero.

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ Use quadratic formula to solve for x.

$x=\dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(-k)}}{2(1)}$

$x=\dfrac{1\pm\sqrt{1+4k}}{2}$ Since a negative number cannot be square rooted, the negative exact value is an extraneous root because it doesn't work.

$x=\dfrac{1+\sqrt{1+4k}}{2}$ Only the positive value is accepted.

Discovery 3a:

$$x=\sqrt{k+\sqrt{k+\ldots}}=\frac{1+\sqrt{1+4k}}{2}$$

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As shown in *Discovery 2b* page 9, an infinite surd can be an integer. Since $\frac{1+\sqrt{1+4k}}{2}$ represents the general exact form for all infinite surds, we can throw in values for k to find values that make the expression an integer.

Example 4:

$$\frac{1+\sqrt{1+4k}}{2}$$

a) $k = 2$ $\quad \frac{1+\sqrt{1+4\ 2}}{2} = \frac{1+\sqrt{1+8}}{2} = \frac{1+\sqrt{9}}{2} = \frac{1+3}{2} = \frac{4}{2} = 2 = Z^+$

b) $k = 5$ $\quad \frac{1+\sqrt{1+4\ 5}}{2} = \frac{1+\sqrt{1+20}}{2} = \frac{1+\sqrt{21}}{2} \neq Z^+$

c) $k = 8$ $\quad \frac{1+\sqrt{1+4\ 8}}{2} = \frac{1+\sqrt{1+32}}{2} = \frac{1+\sqrt{33}}{2} \neq Z^+$

d) $k = 12$ $\quad \frac{1+\sqrt{1+4\ 12}}{2} = \frac{1+\sqrt{1+48}}{2} = \frac{1+\sqrt{49}}{2} = \frac{1+7}{2} = \frac{8}{2} = 4 = Z^+$

e) $k = 16$ $\quad \frac{1+\sqrt{1+4\ 16}}{2} = \frac{1+\sqrt{1+64}}{2} = \frac{1+\sqrt{65}}{2} \neq Z^+$

f) $k = 20$ $\quad \frac{1+\sqrt{1+4\ 20}}{2} = \frac{1+\sqrt{1+80}}{2} = \frac{1+\sqrt{81}}{2} = \frac{1+9}{2} = \frac{10}{2} = 5 = Z^+$

$x = 2, 12, 20$

From this it is shown that $1+4k = N^2$, from this we want to again solve for k to see what value k can be.

$$k = \frac{N^2-1}{4}$$

Since k is always an even number as seen in *Example 4*, $N^2 - 1$ must be an even number because an even number divided an even number still makes an even number(k). So:

$N^2 - 1$	= an even number.
N^2	= an odd number.
N	= an odd number because the square root of an odd number always equals an odd number.
$N = ab + 1$	a is any even natural number because the product of any number multiplied with an even number is even, and b is any natural number.
$N = 2b + 1$	substitute a with 2 since it is the lowest even natural number.

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Now that we know what N is, we can substitute $N = 2b + 1$ into $k = \frac{N^2 - 1}{4}$ making:

$$k = \frac{(2b+1)^2 - 1}{4}$$
$$k = \frac{(4b^2 + 4b + 1) - 1}{4}$$
$$k = \frac{(4b^2 + 4b + 1) - 1}{4}$$
$$k = \frac{(4b^2 + 4b)}{4}$$
$$k = b^2 + b$$
$$k = b(b+1)$$

Discovery 4a:

$k = b(b+1)$ k is the product of two consecutive numbers.

To prove that $k = b(b+1)$, will give $\sqrt{k + \sqrt{k + \dots}}$ an integral value, we can substitute $k = b(b+1)$ into $\frac{1+\sqrt{1+4k}}{2}$.

Example 4a:

$$\frac{1+\sqrt{1+4k}}{2}$$

a) $k = 2(2+1) = 6$ $\quad \frac{1+\sqrt{1+4(6)}}{2} = \frac{1+\sqrt{1+24}}{2} = \frac{1+\sqrt{25}}{2} = \frac{1+5}{2} = \frac{6}{2} = 3 = Z^+$

b) $k = 5(5+1) = 30$ $\quad \frac{1+\sqrt{1+4(30)}}{2} = \frac{1+\sqrt{1+120}}{2} = \frac{1+\sqrt{121}}{2} = \frac{1+11}{2} = \frac{12}{2} = 6 = Z^+$

c) $k = 0.5(0.5+1) = 0.75$ $\quad \frac{1+\sqrt{1+4(0.75)}}{2} = \frac{1+\sqrt{1+3}}{2} = \frac{1+\sqrt{4}}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5 = Q$

d) $k = 1.5(1.5+1) = 3.75$ $\quad \frac{1+\sqrt{1+4(3.75)}}{2} = \frac{1+\sqrt{1+15}}{2} = \frac{1+\sqrt{16}}{2} = \frac{1+4}{2} = \frac{5}{2} = 2.5 = Q$

Discovery 4b:

$\frac{1+\sqrt{1+4k}}{2} = (b+1)$ When the infinite surd is an integral value, the integral value is equal to the greater consecutive factor of *k*.

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The limitation to the general statement of the infinite surd $\frac{1+\sqrt{1+4k}}{2}$ being expressed is an integral value is that in $k = b(b+1)$, b must be any positive integral value equal or above zero. As shown in *Example 4a*, if b is a rational number, then $\frac{1+\sqrt{1+4k}}{2}$ equals a rational number.

Discovery 4c:

Limit to $(b+1)$: $b = Z^{\geq}$

As we have seen from *Examples 1* and *2*, *infinite surds do not increase forever, their curve actually levels out as the terms in the infinite surd increases*. We also saw that in the sequence of an infinite surd, the *next term is the square root of the numerical value in the surd added with the previous term in the sequence.* From *Example 3*, we learned that the general form for the exact value of all infinite surds is $\frac{1+\sqrt{1+4k}}{2}$ and *Example 4* showed that the *exact value can be an integer if* $k = b(b+1)$. In *Example 4* we also learned that $b = Z^{\geq}$ *because if b were a rational number, the exact value of the infinite surd would not be integral.*

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