Mathematics SL
Portifolie

Infinite Surds

An infinite surd is a never-ending positive irrational number. It is a number that can only be expressed exactly using the root sign $\sqrt{\ }$.

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

This sequence above is known as an infinite surd and can be expressed in the terms of a_n:

$a_1 = \sqrt{1+\sqrt{1}} = 1.414213$

$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}} = 1.553773$

$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} = 1.598053$

$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}} = 1.611847$

$a_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}} = 1.616121$

$a_6 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}} = 1.617442$

$a_7 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}} = 1.617851$

$a_8 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}} = 1.617977$

$a_9 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}} = 1.618016$

$$a_{10} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}}} = 1{,}618028$$

etc.

This is the first ten terms and the formula for these sequences is:

$$a_{n+1} = \sqrt{1+a_n}$$

because if we use the term a_2 as an example, this could be proven as:

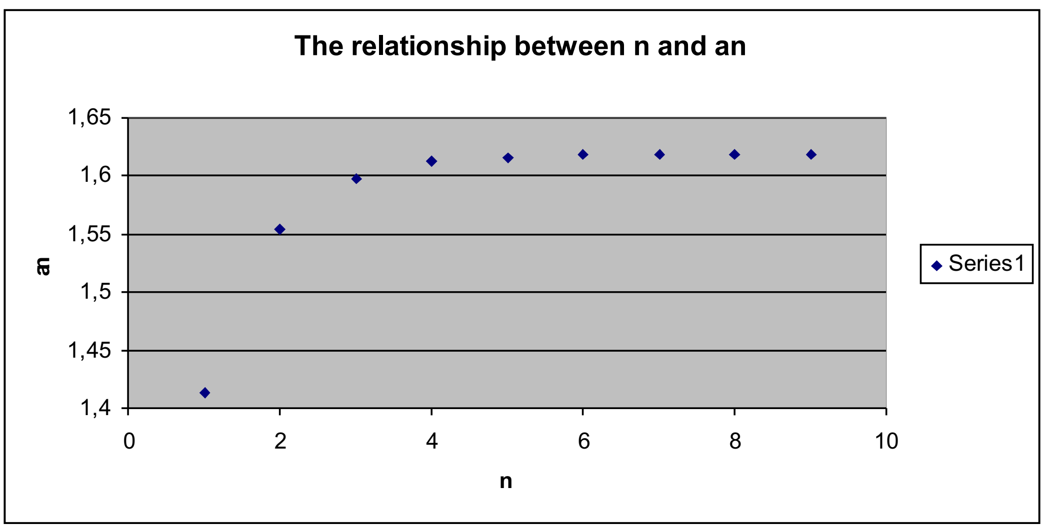
$$a_{n+1} = \sqrt{1+a_n}$$

$$a_{1+1} = \sqrt{1+a_1}$$

$$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}}$$

$a_2 = 1.553773$

By plotting a graph of the ten first term of this sequence the relationship between n and a_n could be shown:



As can be seen from this graph is that the values increase, but then flattens out. The values of a_n moves towards the value of 1.618 approximately, but will actually never reach it. This can be understood by:

$a_n - a_{n+1}$

as n gets very large.

$\lim(a_n - a_{n+1}) \rightarrow 0$

When n gets very large and approaches infinity, the value approaches 0 because the difference between these two values become very small.

This is a way to find the exact value of this infinite surd:

a_{n+1} can also be written as a_n so therefore

$a_n = \sqrt{1+a_n}$

$a_n^2 = 1+a_n$

$a_n^2 - a_n - 1 = 0$

abc- formula:

$$a = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times1\times(-1)}}{2\times1}$$

$a_1 = 1.618$

$a_2 = -0.618$

Since the value has to be a positive number, the exact value of this infinite surd is 1.618.

To research this furthermore, another example of an infinite surd can be considered:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2\ldots}}}}}$$

Considering the infinite surd above, the first ten terms is:

$b_1 = \sqrt{2+\sqrt{2}} = 1.847759065$

$b_2 = \sqrt{2+\sqrt{2+\sqrt{2}}} = 1.961570561$

$b_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = 1.990369453$

$b_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} = 1.997590912$

$b_5 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}} = 1.999397637$

$b_6 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} = 1.999849404$

$b_7 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}} = 1.999962351$

$b_8 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}} = 1.999990588$

$b_9 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}} = 1.999997647$

$b_{10} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}} = 1,999999412$

By replacing a_n by b_n the formula for this sequence is:

$b_{n+1} = \sqrt{2+b_n}$

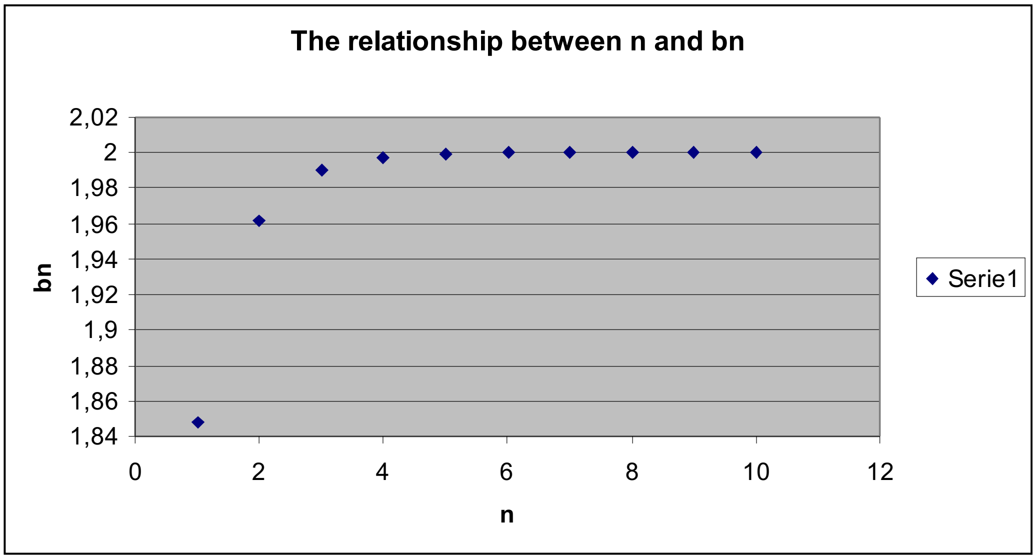
because if we use b_2 as an example, the answer will be:

$b_{1+1} = \sqrt{2+b_1}$

$b_2 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

$b_2 = 1.961570561$

By plotting a graph, the relationship between n and b_n can be shown:



As can be seen from this graph is that the value of b_n increases as n gets larger. After a while the curve flattens out, but continue to rise. The values of bn approaches 2, but will actually never reach it. This can be understood by:

$b_n - b_{n+1}$

As n becomes very large, the value approaches 0 because:

$\lim(b_n - b_{n+1}) \to 0$

When n becomes very large and reaches infinity, the value approaches 0. This is because the difference between the two values is so small that the difference becomes insignificant.

The exact value of this infinite surd is:

b_{n+1} can also be written as b_n so:

$b_n = \sqrt{2 + b_n}$

$b_n^2 = 2 + b_n$

$b_n^2 - b_n - 2 = 0$

abc- formula

$$b = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$

$b_1 = 2$

$b_2 = -1$

Since the value of b_n can only be a positive number the exact value of this infinite surd will be 2.

Now, considering a general term

$$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\ldots}}}}$$

were the first term is:

$$\sqrt{k+\sqrt{k}}$$

The expression for the exact value of this general infinite surd in terms of *k* will be:

$a_n^2 = k + a_n$

$a_n^2 - a_n - k = 0$

abc formula

$$a_n = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times 1\times(-1)}}{2\times 1}$$

$$a_n = \frac{1+\sqrt{1+4k}}{2}$$

Above you can see the expression for the exact value for the general term because whatever number we use instead of *k*, we will find the exact value for the specific term. One example that will make the expression an integer is:

$k = 2$

$$a_n = \frac{1+\sqrt{1+4\times 2}}{2}$$

$$a_n = \frac{1+3}{2}$$

$a_n = 2$

Another example is:

$k = 6$

$a_{\mathrm{n}} = \dfrac{1+\sqrt{1+4\times 6}}{2}$

$a_{\mathrm{n}} = \dfrac{1+5}{2}$

$a_{\mathrm{n}} = 3$

These values show that the answers of these values are integers. The general statement represented by k is:

$a_n = \dfrac{1+\sqrt{1+4k}}{2}$

$2a_{\mathrm{n}} = 1+\sqrt{1+4k}$

$(2a_{\mathrm{n}}\text{-}1)^2 = 1+4k$

$\dfrac{(2a_n-1)^2-1}{4} = k$

The expression above gives the solutions possible, with the answer expressed as an integer, for the possible k- values.

Testing the validity of the general statement using other values of k:

<u>Ex. 1</u>

$k = \text{-}2$

$a_n = \dfrac{1+\sqrt{1+4\times(-2)}}{2}$

$a_{\mathrm{n}} = \dfrac{1+\sqrt{-7}}{2}$ No solution

<u>Ex. 2</u>

$k = 0$

$a_{\mathrm{n}} = \dfrac{1+\sqrt{1+4\times 0}}{2}$

$a_{\mathrm{n}} = \dfrac{1+1}{2}$

$a_{\mathrm{n}} = 1$

Ex. 3

$k = 1.5$

$$a_n = \frac{1 + \sqrt{1 + 4 \times 1.5}}{2}$$

$$a_n = \frac{1 + 2.64575}{2}$$

$a_n = 1.82288$

Ex. 4

$k = 2$

$$a_n = \frac{1 + \sqrt{1 + 4 \times 2}}{2}$$

$$a_n = \frac{1 + 3}{2}$$

$a_n = 2$

Ex. 5

$k = 2.1$

$$a_n = \frac{1 + \sqrt{1 + 4 \times 2.1}}{2}$$

$$a_n = \frac{1 + 3.06594}{2}$$

$a_n = 2.03297$

Ex. 6

$k = 2.5$

$$a_n = \frac{1 + \sqrt{1 + 4 \times 2.5}}{2}$$

$$a_n = \frac{1 + 3.31663}{2}$$

$a_n = 2.15831$

As can be seen from example number 1 is that if *k* is a negative number we get square root of a negative number and therefore there will be no solution. Because of this, *k* cannot be a negative number.

From example 2 and 4 this can be found: There are few numbers that gives the answer of an integer in this sequence expressed with *k*. Some of the values are the numbers 0, 2, 6, 12 and 20 etc. From this we can see an increasing trend were:

$0 = 2 \times 0$
$2 = 2 \times 1$
$6 = 2 \times 3$
$12 = 2 \times 6$
$20 = 2 \times 10$

We can see that from 0 to 2 the number we multiply by the constant, 2, increases by 1, and from 2 to 6 it increases by 2. From 6 to 12 it increases by 3, and from 12 to 20 it increases by 4.

If we use other numbers from 0 to 20 than those above, we get decimal numbers. This can be seen from example 3, 5 and 6. As *k* increases, the value of a_n increases as well.

The scope is $0 < n > \infty$, because n cannot be a negative number.