

Infinite Surds

➤ Introduction to Infinite Surds

● Definition of a surd

- An irrational number whose exact value can only be expressed using the radical or root symbol is called a surd.

E.g.) $\sqrt{2}$ is a surd, because the square root of two is irrational.

● The origin of the word

- In or around 825AD, Al-Khwarizmi who was an Arabic mathematician during the Islamic empire referred to the rational numbers as 'audible' and irrational as 'inaudible'. Then, the European mathematician, Gherardo of Cremona, adopted the terminology of surds (*surdus* means 'deaf' or 'mute' in Latin) in 1150. In English language, the 'surd' appeared in the work of Robert Recorde's *The Pathway to Knowledge*, published in 1551.

● The symbol, use of the word radical

- The radical symbol $\sqrt{\quad}$ depicts surds, with the upper line above the expression called the vinculum. Also, a cube root takes the form $\sqrt[3]{a}$, which corresponds to $a^{1/3}$ when expressed using indices. So, all roots can remain in surd form.

● A definition of infinite surds

- An infinite surd is a never ending irrational number and its exact value would be left in square root form.

E.g.) The general infinite surd $a_n = \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}$

Therefore, $a_3 = \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}$

- The following expression is an example of an infinite surd.

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

Consider this surd as a sequence of terms a_n where:

$$a_1 = \sqrt{1 + \sqrt{1}}$$

$$a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$$

$$a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} \quad \text{and so on.}$$

- Find a formula for a_{n+1} in terms of a_n .

Since $a_1 = \sqrt{1 + \sqrt{1}}$ and $a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$,

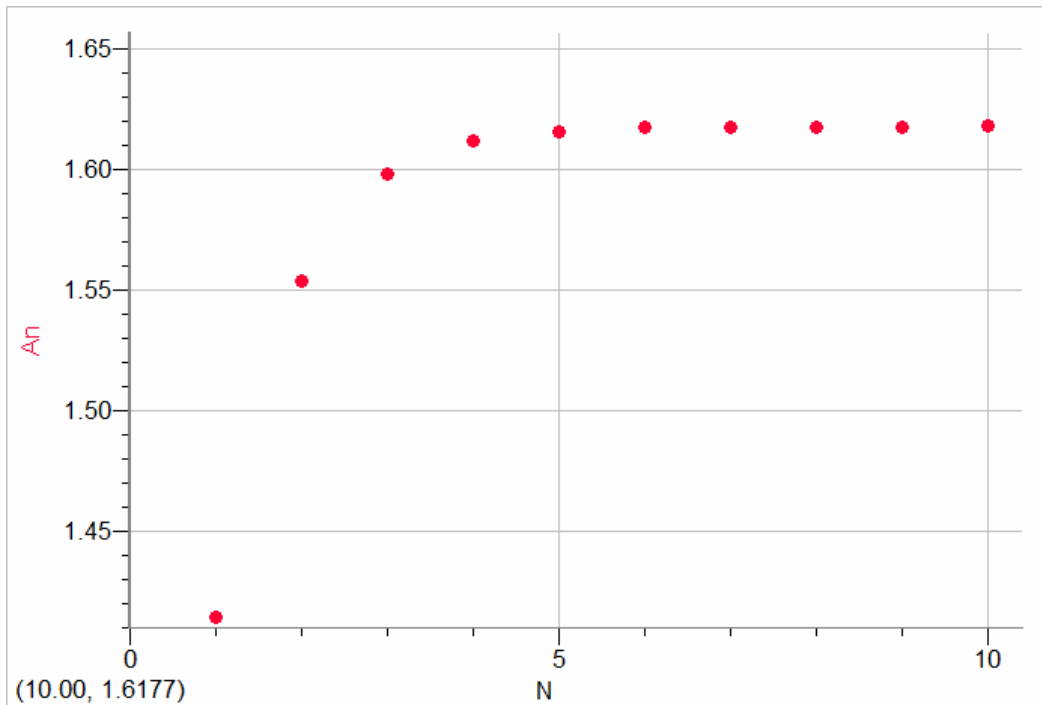
a_2 can be written as $\sqrt{1 + a_1}$

Therefore, $a_{n+1} = \sqrt{1 + a_n}$

- The first ten terms of the sequence

Term	Surd	Decimal
a_1	$\sqrt{1 + \sqrt{1}}$	1.414213562
a_2	$\sqrt{1 + \sqrt{1 + \sqrt{1}}}$	1.553773974
a_3	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}$	1.598053182
a_4	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}$	1.611847754
a_5	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}$	1.616121207
a_6	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}$	1.617442799
a_7	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}$	1.617851291
a_8	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}}$	1.617977531
a_9	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}}}$	1.618016542
a_{10}	$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}}}}$	1.618028597

- Plot the relation between n and a_n



→ I can see from the graph above that the value of a_n approaches approximately 1.618 but never reach it. Also, I can suggest from the chart below that the consecutive differences are rapidly approaching zero as n gets larger.

- What does this suggest about the value of $a_{n+1} - a_n$ as n gets very large?

Terms	Difference
$a_2 - a_1$	0.139560412
$a_3 - a_2$	0.044279208
$a_4 - a_3$	0.013794572
$a_5 - a_4$	0.004273453
$a_6 - a_5$	0.001321592
$a_7 - a_6$	0.000408492
$a_8 - a_7$	0.0002362240
$a_9 - a_8$	0.000039011
$a_{10} - a_9$	0.000012055

- Use your results to find the exact value for this infinite surd

Let a_n be S

$$S = \sqrt{1 + S}$$

$$S^2 = 1 + S$$

$$S^2 - S - 1 = 0$$

Use quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Therefore, $S = \frac{1 \pm \sqrt{5}}{2}$

The value must be positive since the graph showed that there are no negative values.

So, disregard the negative sign.

Thus, the exact value is $\frac{1 + \sqrt{5}}{2}$

- Consider another infinite surd $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}$ where the first term is $\sqrt{2+\sqrt{2}}$.

$$A_1 = \sqrt{2+\sqrt{2}}$$

$$A_2 = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$A_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \quad \text{and so on.}$$

- Find a formula for a_{n+1} in terms of a_n .

Since $a_1 = \sqrt{2+\sqrt{2}}$ and $a_2 = \sqrt{2+\sqrt{2+\sqrt{2}}}$,

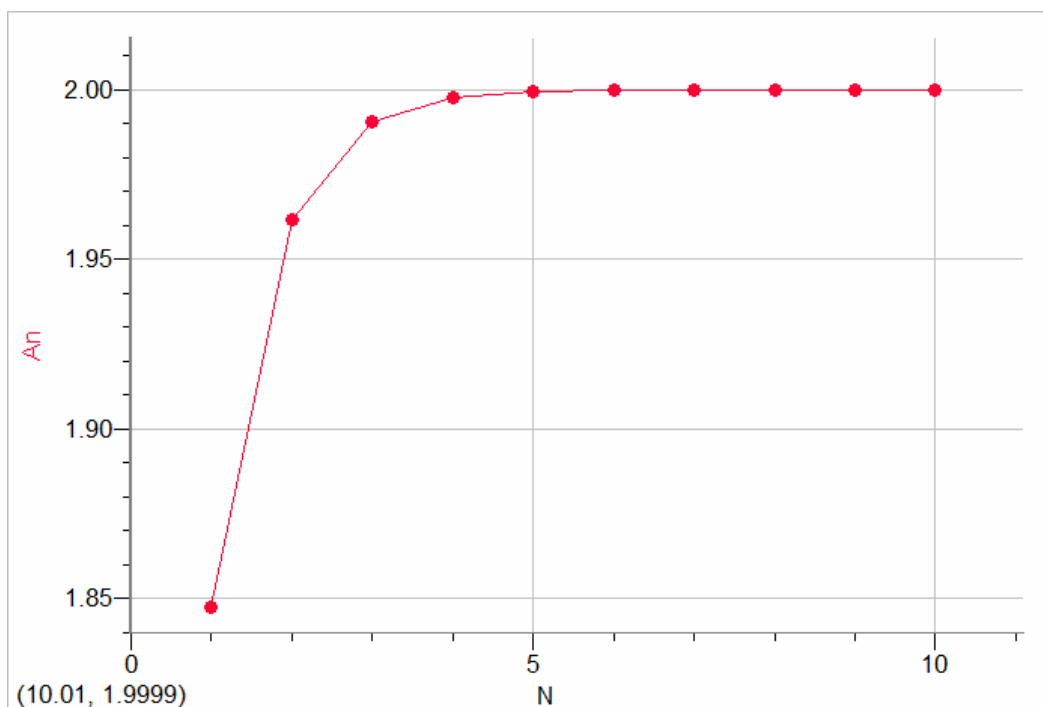
A_2 can be written as $\sqrt{2+a_1}$

Therefore, $a_{n+1} = \sqrt{2+a_n}$

- The first ten terms of the sequence.

Term	Surd	Decimal
a_1	$\sqrt{2+\sqrt{2}}$	1.847759065
a_2	$\sqrt{2+\sqrt{2+\sqrt{2}}}$	1.961570561
a_3	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$	1.990369453
a_4	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$	1.997590912
a_5	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$	1.999397637
a_6	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}$	1.9998494404
a_7	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}$	1.9999849404
a_8	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}$	1.999990588
a_9	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}$	1.999997647
a_{10}	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}}$	1.999999412

- Plot the relation between n and a_n



→ I can see easily from the graph that the value of a_n approaches 2 but never reach it and the graph becomes less steep and the value of a_n does not increase as the value of n increases.

- What does this suggest about the value of $a_{n+1} - a_n$ as n gets very large?

Terms	Difference
$a_2 - a_1$	0.113811496
$a_3 - a_2$	0.028798892
$a_4 - a_3$	0.007221459
$a_5 - a_4$	0.001806725
$a_6 - a_5$	0.0004518034
$a_7 - a_6$	0.0001355
$a_8 - a_7$	0.0000056476
$a_9 - a_8$	0.000007059
$a_{10} - a_9$	0.000001765

→ As you can see from the table on the left, the differences between a_{n+1} and a_n become smaller and rapidly approaching zero as n gets larger.

● **Find the exact value for this infinite surd.**

* Let a_n be X

$$X = \sqrt{2 + X}$$

$$X^2 = 2 + X$$

$$X^2 - X - 2 = 0$$

Use quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\text{Therefore, } X = \frac{1 \pm \sqrt{1+8}}{2}$$

The value must be positive since the graph showed that there are no negative values, so disregard the negative sign.

$$\text{Thus, } X = \frac{1 + \sqrt{9}}{2} = 2$$

➤ **Finding an expression for the exact value of following general infinite surd in terms k .**

The general infinite surd $\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}$ where the first term is $\sqrt{k + \sqrt{k}}$.

➤ **Consider this surd as a sequence of terms b_n**

$$B_1 = \sqrt{K + \sqrt{K}}$$

$$B_2 = \sqrt{K + \sqrt{K + \sqrt{K}}}$$

$$B_3 = \sqrt{K + \sqrt{K + \sqrt{K + \sqrt{K}}}} \text{ and so on.}$$

● **Find a formula for b_{n+1} in terms of b_n .**

Since $b_1 = \sqrt{k + \sqrt{k}}$ and $b_2 = \sqrt{k + \sqrt{k + \sqrt{k}}}$,

B_2 can be written as $\sqrt{K + b_1}$

Therefore, $b_{n+1} = \sqrt{k + b_n}$

● The first ten terms of the sequence

Term	Surd
b_1	$\sqrt{k + \sqrt{k}}$
b_2	$\sqrt{k + \sqrt{k + \sqrt{k}}}$
b_3	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}$
b_4	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}$
b_5	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}}$
b_6	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}}}$
b_7	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}}}}$
b_8	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}}}}}$
b_9	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}}}}}}$
b_{10}	$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k}}}}}}}}}}}$

→ From the calculation for the differences between a_{n+1} and a_n above, I can know that the difference between a_{n+1} and a_n become smaller and rapidly approaching zero as n gets larger.
Therefore, $b_{n+1} - b_n = 0$

● Find the expression for the exact value of this general infinite surd.

$$b_{n+1} - b_n = 0$$

$$b_{n+1} = b_n$$

Substitute $\sqrt{k + b_n}$ for b_{n+1} because the formula for $b_{n+1} = \sqrt{k + b_n}$

$$\text{So, } \sqrt{k + b_n} = b_n$$

* Let b_n be X

$$X = \sqrt{k + X}$$

$$X^2 = k + X$$

$$X^2 - X - k = 0$$

Use the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$X = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-k)}}{2(1)} = \frac{1 \pm \sqrt{1 + 4k}}{2}$$

➤ **The value of an infinite surd is not always an integer.**

● **Find some values of k that make the expression an integer.**

So, it is known that the expression for the exact value of $\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}$

$$= \frac{1 \pm \sqrt{1 + 4K}}{2} \quad (X = \frac{1 \pm \sqrt{1 + 4K}}{2})$$

$$X = \frac{1 \pm \sqrt{1 + 4K}}{2}$$

If X is an integer, the numerator has to be even because it is divided by 2. So

$\sqrt{1 + 4k}$ has to be odd which means that $1 + 4k$ has to be also odd.

Therefore, $1 + 4k$ has to be a perfect square so that $\sqrt{1 + 4k}$ is an integer.

Example 1) If I put 12 into the k ,

$$X = \frac{1 + \sqrt{1 + 48}}{2} = 4$$

Example 2) If I put 2 into the k ,

$$X = \frac{1 + \sqrt{9}}{2} = 2$$

Example 3) If I put 6 into the k ,

$$X = \frac{1 - \sqrt{25}}{2} = -2$$

So, some values of k that make the expression an integer are 2, 6 and 12.

● **Find the general statement that represents all the values of k for which the expression is an integer.**

Since $\sqrt{1 + 4k}$ is an odd perfect square, **let m be any odd number**

Hence, $1 + 4k = (m)^2$ because the square of an odd number is also an odd number.

$$1 + 4k = m^2$$

$$4k = m^2 - 1$$

$$\text{Thus, } k = \frac{m^2 - 1}{4}$$

➤ **Test the validity of general statement using other values of k**

The general statement: $k = \frac{m^2 - 1}{4}$

Testing 1) If $k = -5$

$$-5 = \frac{m^2 - 1}{4}$$

$$-20 = m^2 - 1$$

$$m^2 = -19$$

So, there is no value of m .

Thus, k cannot be negative.

Testing 2) If $k = \frac{3}{4}$

$$\frac{3}{4} = \frac{m^2 - 1}{4}$$

$$3 = m^2 - 1$$

$$m^2 = 4$$

$$m = 2$$

$$\text{Since } X = \frac{1 \pm \sqrt{1 + 4k}}{2},$$

$$X = \frac{1 \pm \sqrt{4}}{2} = \frac{3}{2} \text{ or } -\frac{1}{2} \quad \text{So, } X \text{ is not an integer.}$$

Thus, k cannot be a fraction.

Testing 3) If $k = 0$

$$X = \frac{1 \pm 1}{2} = 1 \text{ or } 0 \quad \text{So, } X \text{ is an integer.}$$

Thus, k has to be an integer that is greater or equal to 0.

● **Discuss the scope and/or limitations of your general statement.**

- According to the three tests above, I found that k cannot be a negative number and a fraction. If k is a negative number and a fraction, X , which expresses the exact value of the general infinite surd,

$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}$ cannot be an integer. It becomes an integer only when k is an integer that is greater or equal to 0.

● **Explain how you arrived at your general statement.**

- First, I found a formula for the general infinite surd $\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}$ where the first term is $\sqrt{k + \sqrt{k}}$. I considered this surd as a sequence of terms b_n , so I could find that $b_{n+1} = \sqrt{k + b_n}$. Then, I could know from the calculation for the differences between a_{n+1} and a_n that the differences between b_{n+1} and b_n become smaller and rapidly approaching zero as n gets larger. After that, I substituted $\sqrt{k + b_n}$ for b_{n+1} because the formula for $b_{n+1} = \sqrt{k + b_n}$ and I got $\sqrt{k + b_n} = b_n$. I let b_n be X , so I got $X^2 - X - K = 0$ at the end. And then I used the quadratic formula to find the expression for the exact value of this general infinite surd which is $\frac{1 \pm \sqrt{1 + 4K}}{2}$. After I got the expression for the exact value, I tried to find some values of k that make the expression an integer and I could realize that if X is an integer, the numerator has to be even because it is divided by 2. So, I found that $\sqrt{1 + 4k}$ has to be an odd perfect square to X be an integer. Since I found out this fact, I let m be any odd number thus, $1 + 4k = (m)^2$ because the square of an odd number is also an odd number. Finally, I developed the equation and I arrived at my general statement which is $k = \frac{m^2 - 1}{4}$.