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Infinite Surds Coursework

The following expression is an example of an infinite surd.

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

Find the formula for a_{n+1} in terms of a

$a_1 = \sqrt{1+\sqrt{1}}$

$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}}$ $\qquad a_2 = \sqrt{1+a_1}$

$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}$ $\qquad a_3 = \sqrt{1+a_2}$

$a_{n+1} = \sqrt{1+a_n}$

$a_n = \sqrt{1+a_{n-1}}$

Calculate the decimal values of the first ten terms of the sequence

a_1 = 1.414213562373100
a_2 = 1.553773974030040
a_3 = 1.598053182478620
a_4 = 1.611847754125250
a_5 = 1.616121206508120
a_6 = 1.617442798527390
a_7 = 1.617851290609670
a_8 = 1.617977530934740
a_9 = 1.618016542231490
a_{10} = 1.618028597470230

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Using technology, plot the relation between n and a_n. Describe what you notice.

n	a_n
1	1.414213562373100
2	1.553773974030040
3	1.598053182478620
4	1.611847754125250
5	1.616121206508120
6	1.617442798527390
7	1.617851290609670
8	1.617977530934740
9	1.618016542231490
10	1.618028597470230
11	1.618032322752000
12	1.618033473928150
13	1.618033829661220
14	1.618033939588790
15	1.618033973558280
16	1.618033984055430
17	1.618033987299220
18	1.618033988301610
19	1.618033988611370
20	1.618033988707090
21	1.618033988736670
22	1.618033988745810
23	1.618033988748630
24	1.618033988749500
25	1.618033988749770
26	1.618033988749860
27	1.618033988749880
28	1.618033988749890
29	1.618033988749890
30	1.618033988749890
31	1.618033988749890
32	1.618033988749890
33	1.618033988749890
34	1.618033988749890
35	1.618033988749890
36	1.618033988749890
37	1.618033988749890
38	1.618033988749890
39	1.618033988749890
40	1.618033988749890

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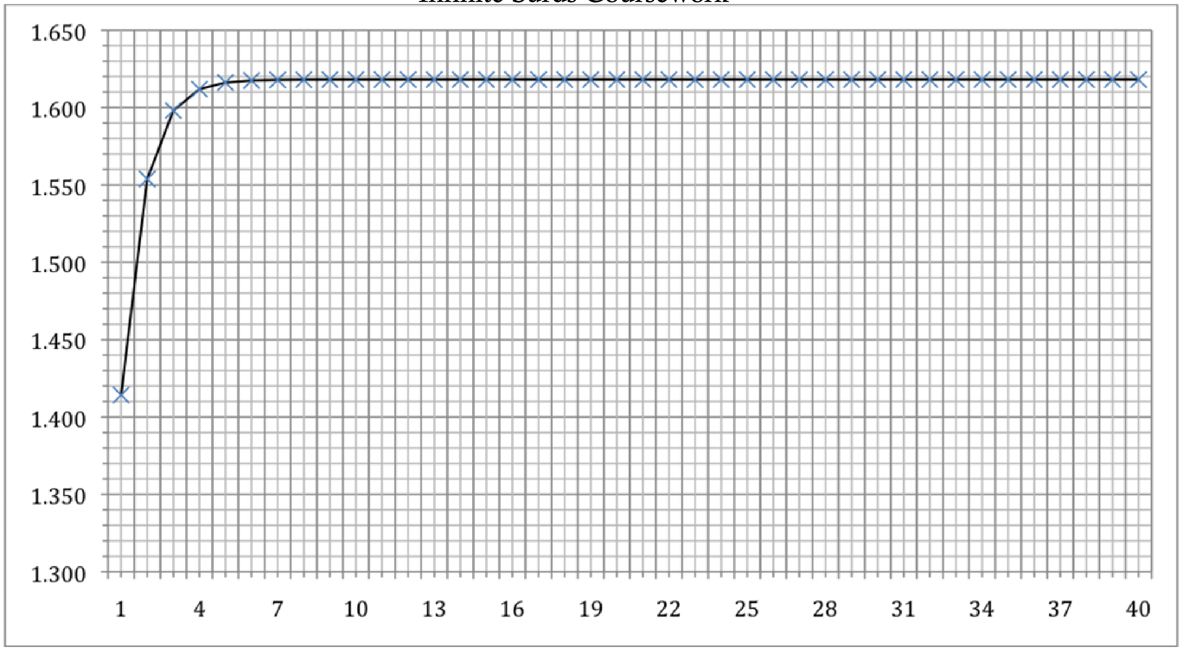

By plotting the relation between n and a_n, one notices that as n increases, a_n increases. However this increase is not proportional to the increase of n, a_n seems to be increasing towards 1.62.
Once n reaches 28 a_n ceases to increase, remaining stable at 1.618033988749890.

This suggests that as n becomes very large $a_n - a_{n+1} = 0$

As such, we can conclude that the exact value for this infinite surd is 1.618033988749890.

Consider another infinite surd:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}}$$

Find the formula for a_{n+1} in terms of a

$a_1 = \sqrt{2+\sqrt{2}}$

$a_2 = \sqrt{2+\sqrt{2+\sqrt{2}}}$ $\qquad a_2 = \sqrt{2+a_1}$

$a_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$ $\qquad a_3 = \sqrt{2+a_2}$

$a_{n+1} = \sqrt{2+a_n}$

$a_n = \sqrt{2+a_{n-1}}$

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Calculate the decimal values of the first ten terms of the sequence

a_1 = 1.847759065
a_2 = 1.9615705608
a_3 = 1.9903694533
a_4 = 1.9975909124
a_5 = 1.9993976374
a_6 = 1.9998494037
a_7 = 1.9999623506
a_8 = 1.9999905876
a_9 = 1.9999976469
a_{10} = 1.9999994117

Using technology, plot the relation between n and a_n. Describe what you notice.

n	a_n
1	1.847759065
2	1.9615705608
3	1.9903694533
4	1.9975909124
5	1.9993976374
6	1.9998494037
7	1.9999623506
8	1.9999905876
9	1.9999976469
10	1.9999994117
11	1.9999998529
12	1.9999999632
13	1.9999999908
14	1.9999999977
15	1.9999999994
16	1.9999999999
17	2.0000000000
18	2.0000000000
19	2.0000000000
20	2.0000000000

21	2.0000000000
22	2.0000000000
23	2.0000000000
24	2.0000000000
25	2.0000000000
26	2.0000000000
27	2.0000000000
28	2.0000000000
29	2.0000000000
30	2.0000000000
31	2.0000000000
32	2.0000000000
33	2.0000000000
34	2.0000000000
35	2.0000000000
36	2.0000000000
37	2.0000000000
38	2.0000000000
39	2.0000000000
40	2.0000000000

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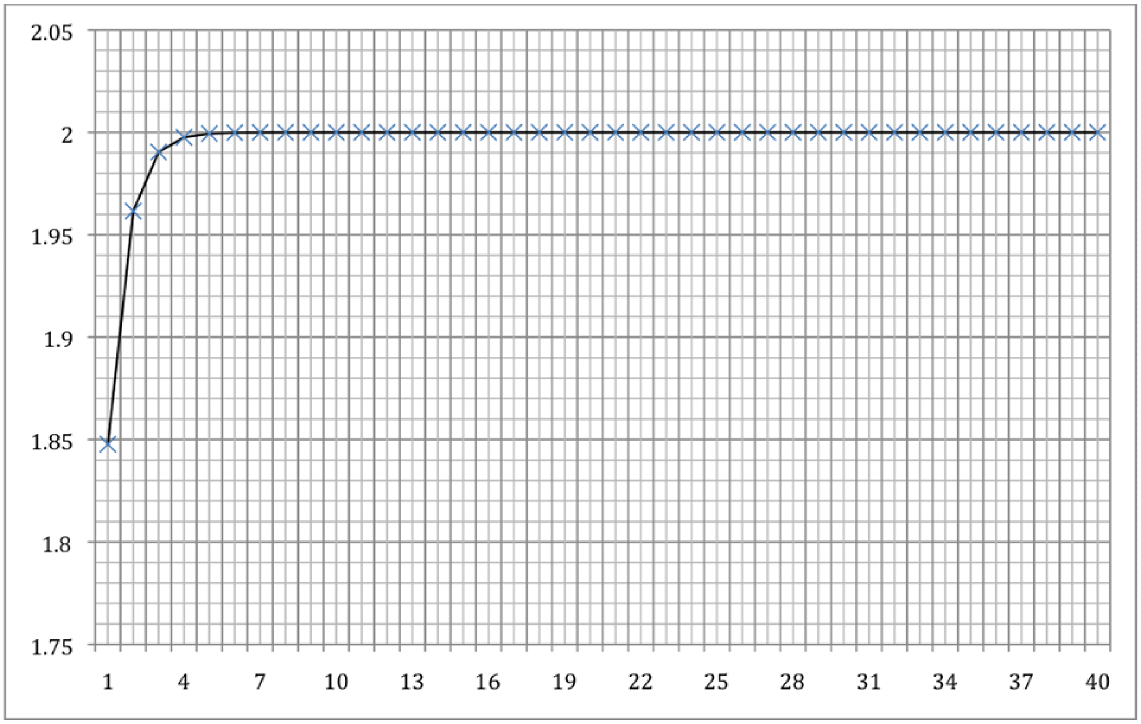

By plotting the relation between n and a_n, one notices that as n increases, a_n increases. However this increase is not proportional to the increase of n, a_n seems to be increasing towards 2.
Once n reaches 17 a_n ceases to increase, remaining stable at 2.

This suggests that as n becomes very large $a_n - a_{n+1} = 0$

As such, we can conclude that the exact value for this infinite surd is 2.

Consider the general infinite surd :

$$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\ldots}}}}}$$

Find the formula for a_{n+1} in terms of a

$a_1 = \sqrt{k+\sqrt{k}}$

$a_2 = \sqrt{k+\sqrt{k+\sqrt{k}}}$ $\qquad a_2 = \sqrt{k+a_1}$

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$a_3 = \sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k}}}}$ $a_3 = \sqrt{k+a_2}$

$$a_{n+1} = \sqrt{k + a_n}$$

$$a_\infty = \sqrt{k + a_\infty}$$

let $a_\infty = x$

$$x = \sqrt{k + a_\infty}$$

$$x = x^2 - x - k$$

x is a quadratic equation we can therefore find the discriminant Δ

$$\Delta = b^2 - 4ac$$

$$\Delta = x^2 - 4(x^2)(-k)$$

$$\Delta = x^2 + 4x^2k$$

$$\Delta = x^2(1 + 4k)$$

$\Delta > 0$

Since Δ is greater than 0 this quadratic equation has two distinct solutions.

$x_1 = \dfrac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \dfrac{-b+\sqrt{\Delta}}{2a}$

$$x_1 = \frac{x - \sqrt{x^2(1+4k)}}{2x^2}$$

$x_1 < 0$ as an infinite surd can only be positive we can reject this answer.

$$x_2 = \frac{x + \sqrt{x^2(1+4k)}}{2x^2}$$

$$x_2 = \frac{x + \sqrt{x^2(1+4k)}}{2x^2}$$

$$x_2 = \frac{1 + \sqrt{1+4k}}{2}$$

Therefore $a_\infty = \dfrac{1+\sqrt{1+4k}}{2}$

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Some values of *k* that make the expression an integer are: 0, 2, 6, 12, 20

Values of a_∞	Values of $4k$	Values of k
1	0	0
2	8	2
3	24	6
4	48	12
5	80	20
6	120	30
7	168	42
8	224	56
9	288	72
10	360	90
11	440	110
12	528	132
13	624	156
14	728	182

14	728	182
15	840	210
16	960	240
17	1088	272
18	1224	306
19	1368	342
20	1520	380
21	1680	420
22	1848	462
23	2024	506
24	2208	552
25	2400	600
26	2600	650
27	2808	702
28	3024	756

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For a_∞ to be an integer, $\sqrt{1+4k}$ has to be an odd perfect square.

Values of odd perfect squares	57	Values of $\sqrt{1+4k}$	3249
1	59	1	3481
3	61	9	3721
5	63	25	3969
7	65	49	4225
9	67	81	4489
11	69	121	4761
13	71	169	5041
15	73	225	5329
17	75	289	5625
19	77	361	5929
21	79	441	6241
23	81	529	6561
25	83	625	6889
27	85	729	7225
29	87	841	7569
31	89	961	7921
33	91	1089	8281
35	93	1225	8649
37	95	1369	9025
39	97	1521	9409
41	99	1681	9801
43	101	1849	10201
45	103	2025	10609
47	105	2209	11025
49	107	2401	11449
51	109	2601	11881
53	111	2809	12321
55	113	3025	12769

One can notice that the value of the odd perfect square is a series with $a_1 = 1$ and $d = 2$

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$\sqrt{1+4k} = x$ with x any integer

$1+4k = x^2$

$k = \dfrac{x^2 - 1}{4}$

We can notice that when b^2 - 1 is odd $\Rightarrow b^2$ is odd $\Rightarrow b$ is odd

We also know that $\sqrt{1+4k}$ has to be an odd perfect square so that a_∞ can be an integer

Let $x = 2y+1$

$k = \dfrac{(2y+1)^2 - 1}{4}$

$k = y^2 + y$

$k = y(y+1)$

Hence k must be the product of two consecutive integers

Therefore:

$a_\infty = \dfrac{1+\sqrt{1+4k}}{2}$

$a_\infty = \dfrac{1+\sqrt{1+4y(y+1)}}{2}$

$a_\infty = \dfrac{1+2y+1}{2}$

$a_\infty = y+1$

y+1 is always an integer

The general statement that represents all the values of k for which the expression is an integer is $k_n = n(n+1)$ with n any integer.

The general statement $k_n = n(n+1)$ is valid for all integers and therefore has no limitations.

You can see from the steps above the process I took to find the general statement.