Math Portfolio
Infinite Surds

Introduction
a surd is an irrational number that can not be written as a fraction of two integers but can only be expressed using the root sign.
Bellow an example of an infinite surd:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}}$$

This surd can be turned into a set of particular numbers **sequence**:

$$a_1 = \sqrt{1+\sqrt{1}} \approx 1.414213562$$

$$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}} \approx 1.553773974$$

$$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} \approx 1.598053182$$

$$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}} \approx 1.611847754$$

$$a_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}} \approx 1.616121207$$

$$a_6 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}} \approx 1.617442799$$

$$a_7 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}} \approx 1.617851291$$

$$a_8 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}} \approx 1.617977531$$

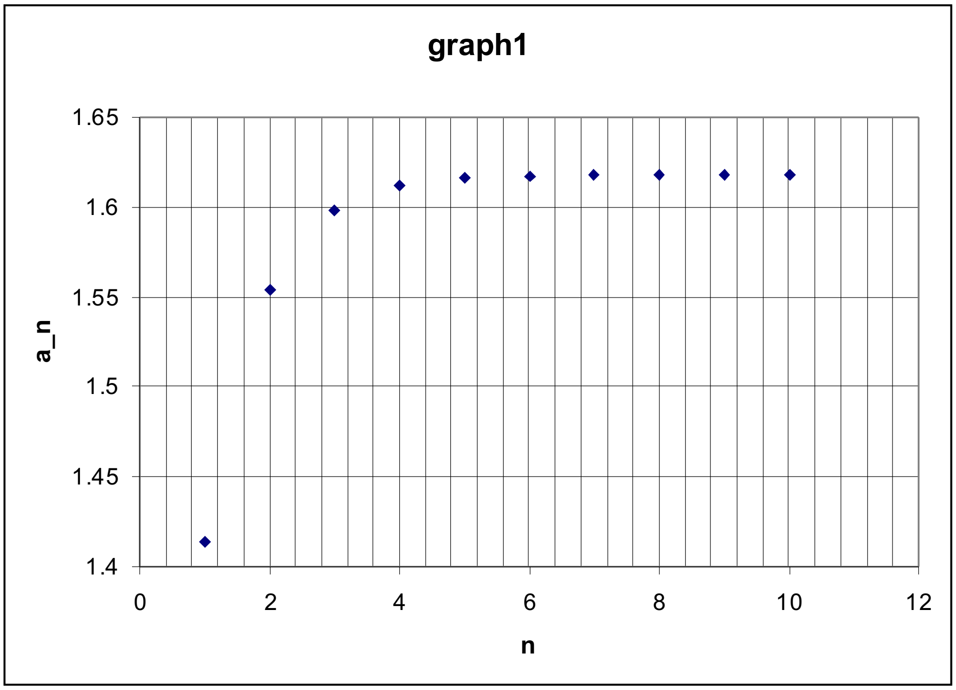
$$a_9 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}} \approx 1.618016542$$

$$a_{10} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}}}$$

$$a_{10} \approx 1.618025897$$

As we can see there are ten terms of this sequence where a_n is the general term of the sequence when $n = 1, a_1$ is the first term of the sequence...Etc.
A formula has been defined for a_{n+1} in terms of a_n:

$$a_{n+1} = \sqrt{1+a_n} \quad (1)$$



A graph has been plotted to show the relation between n and a_n. And it can be oticed that as long as n gets larger, a_n gets closer to a fixed value.

To investigate more about this fixed value we take this equation $a_n - a_{n+1}$ into consideration as n gets bigger.

n	a_n-a_{n+1}
1	- 0.13956
2	-0.04428
3	-0.01380
4	-0.00427
5	-0.00132
6	-0.00040
7	-0.00013
8	-0.00004
9	-0.00001

We can figure out from the table above that when n gets larger, the term (a_n+a_{n+1}) gets closer to zero but it never reaches it

So we can come to the conclusion:

When n approaches infinity, $lim\ (a_n-a_{n+1}) \rightarrow 0$

An expression can be obtained in the case of the relation between n and a_n to get the exact value of the infinite surd:

$$\lim_{n\to\infty} a_n = x$$

If we apply formula (1) to this:

$$x = \sqrt{1+\underset{x}{a_n}} \rightarrow x = \sqrt{1+x}$$

$x^2 = 1 + x \quad \rightarrow \quad x^2 - x - 1 = 0$

The equation can be solved using the solution of a quadric equation:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$$

Whereas $a=1$, $b=-1$, $c=-1$

Two solutions for x were obtained:

x=1.618033989 and x= -0.61803387

The negative value is ignored so x=1.618033989 which is the exact value for this infinite surd.

Another condition of infinite surd can be taken into consideration to acknowledge the point more:

$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2\ldots}}}}$ Where the first term of the sequence is $\sqrt{2+\sqrt{2}}$

The first ten terms of the sequence are:

$$h_1 = \sqrt{2+\sqrt{2}} \approx 1.847759065$$

$$h_2 = \sqrt{2+\sqrt{2+\sqrt{2}}} \approx 1.961570561$$

$$h_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \approx 1.990369453$$

$$h_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \approx 1.997590912$$

$$h_5 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}} \approx 1.999397637$$

$$h_6 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} \approx 1.999849404$$

$$h_7 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}} \approx 1.999962351$$

$$h_8 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}} \approx 1.999990588$$

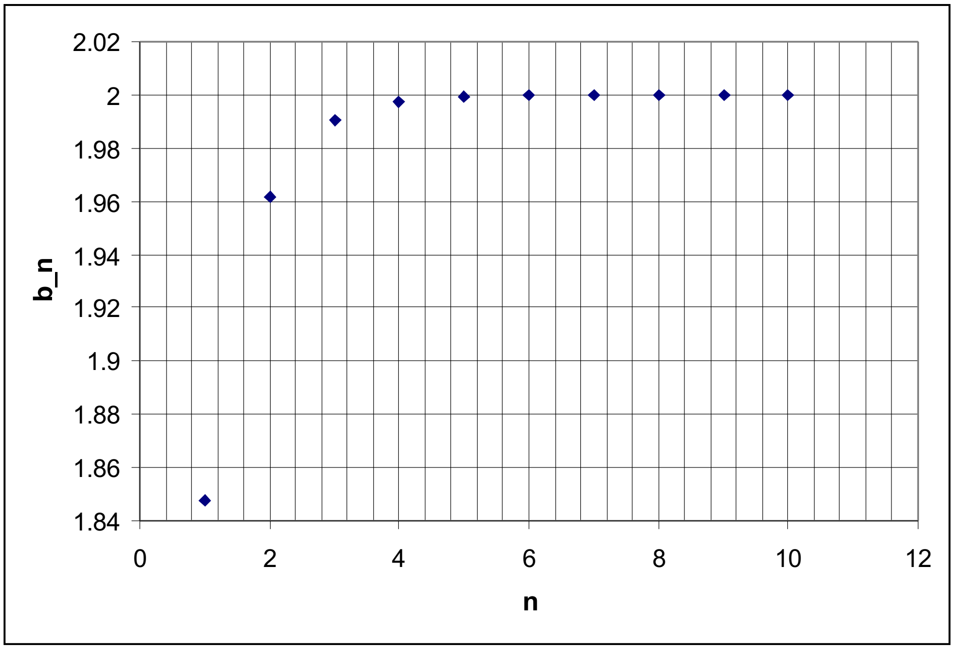
$$h_9 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}} \approx 1.999997647$$

$$h_{10} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}} \approx 1.999999421$$

And the formula of the sequence is obtained according to b_{n+1} which is relative to the term b_n:

$$b_{n+1} = \sqrt{2 + b_n}$$

The graph bellow shows the relation



It can be observed that when n gets bigger, b_n attempt to reach the value 2 which is the exact value for this infinite surd.

$$\lim_{n\to\infty} b_n = 2$$

To prove that 2 is the exact value an expression is used:
Where x= exact value

$$x = \sqrt{2+\underbrace{b_n}_{x}} \Rightarrow x^2 = 2 + x \Rightarrow x^2 - x - 2 = 0$$

$$(x-2)(x+2)=0$$
$$x=2$$
$$x=-2$$

Since only the positive value is concerned then x=2 which is the exact value for the infinite surd.

Now we think about a general infinite surd to prove our previous work.

We consider the general infinite surd as: $\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k\ldots}}}}$

Now let $x = \sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k\ldots}}}}$

Squaring both sides

$$x^2 = k + \underbrace{\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k}}}}}_{x}$$

$x^2=k+x$

$x^2 - x - k=0$ ← the expression for the exact value of the general infinite surd

A general statement could be found to make the expression an integer, and its be solving the equation above using the solution of a quadric equation:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$$

The negative solution is ignored so:

$$x = \frac{1 + \sqrt{1 + 4k}}{2}$$

To find some values of *k* to make the expression an integer:

We can see that 4*k* is an even number and 4*k*+1 is odd, so $\sqrt{1 + 4k}$ is an odd number if 4*k*+1 is a perfect square hence 1+$\sqrt{1 + 4k}$ is an even number and possible to be divided by 2. As a result if 4*k*+1 is a perfect square we can obtain an integral number in the result.

For example let *k*=2:

$$\frac{1 + \sqrt{9}}{2} = 2$$

k=3

$$\frac{1 + \sqrt{13}}{2} \approx 2.303 \quad \leftarrow \text{not integer because 13 is not a perfect square}$$

Thus we come to the conclusion that only limited values of *k* can be used to make the result an integer and those values are any value of *k* can make 4*k*+1 a perfect square such as *k* = 2,6,12…etc.