

MATH PORTFOLIO

Infinite Surds

Introduction

Surds: a surd is a number that can only be expressed exactly using the root sign “ $\sqrt[n]{}$ ” in other words it can be defined as a positive irrational number .

Thus, a number $\sqrt[n]{x}$ is a surd if and only if:

- (a) It is an irrational number
- (b) It is the root of a positive rational number.

The symbol $\sqrt{}$ is called the radical sign. The index **n** is called the order of the surd and **x** is called the radicand.

For example:

$\sqrt{2} = (1.414.....)$ is a surd.

$\sqrt{3} = (1.732.....)$ is a surd.

$\sqrt{4} = (2)$ is not a surd.

$\sqrt{5} = (2.236.....)$ is a surd.

Note: if *n* is a positive integer and *a* be a real number, then if *a* is irrational, $\sqrt[n]{a}$ is not a surd. Again if $\sqrt[n]{a}$ is rational, then also $\sqrt[n]{a}$ is not a surd.

Surds have an infinite number of non-recurring decimal. Hence, surds are irrational numbers and are considered infinite surds.

Following expression is the example of an infinite surd:

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

Considering this surd as a sequence of terms a_n where:

$$a_1 = \sqrt{1 + \sqrt{1}}$$

$$a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$$

$$a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} \text{ etc.}$$

Q: to find a formula for a_{n+1} in terms of a_n

Answer:

$$\rightarrow a_1 = \sqrt{1 + \sqrt{1}} = 1.4142...$$

$$\rightarrow a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} \text{ or } \sqrt{1 + a_1} = 1.5537....$$

$$\rightarrow a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} \text{ or } \sqrt{1 + a_2} = 1.5980....$$

As it is observed that $a_3 = \sqrt{1 + a_2}$; therefore, it can be understood that $a_4 = \sqrt{1 + a_3}$ as the trend has been observed as this until now.

Hence,

$$a_4 = \sqrt{1 + a_3} = 1.6118...$$

$$a_5 = \sqrt{1 + a_4} = 1.6161...$$

$$a_6 = \sqrt{1 + a_5} = 1.6174...$$

$$a_7 = \sqrt{1 + a_6} = 1.6178...$$

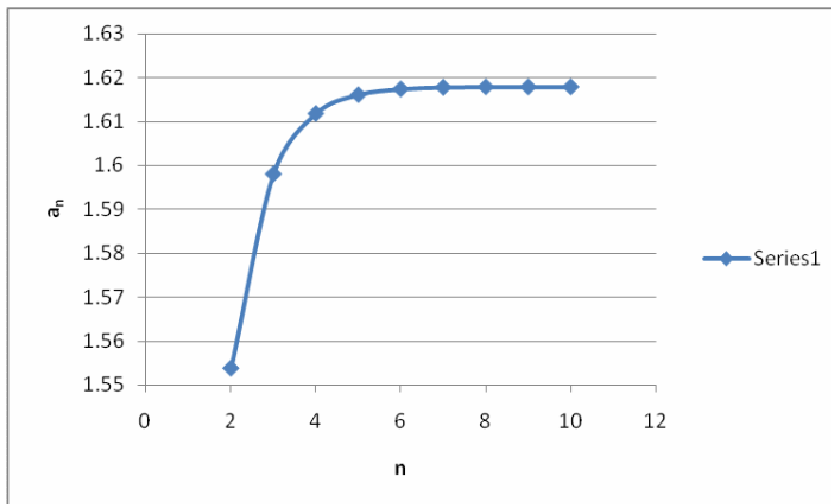
$$a_8 = \sqrt{1 + a_7} = 1.6179...$$

$$a_9 = \sqrt{1 + a_8} = 1.6179...$$

$$a_{10} = \sqrt{1 + a_9} = 1.6179...$$

[NOTE: VALUES OBTAINED USING THE GRAPHICAL DISPLAY CALCULATOR: CASIO fx – 9860G]

n	a_n
1	1.4142
2	1.5537
3	1.5980
4	1.6118
5	1.6161
6	1.6174
7	1.6178
8	1.6179
9	1.6179
10	1.6179



From this graph I understand that after the value keeps on increasing so does the value become similar i.e. 5 onwards the value is almost the same (1.62). the graph as seen faces a substantial rise in the start and the becomes straighter and consistent.

Q: find out the exact value for this surd ?

Answer:

$$a_{n+1} = \sqrt{1 + a_n}$$

As n becomes very large, $a_{n+1} - a_n = 0$ as n is ∞

$$\therefore a_{n+1} = a_n = x.$$

$$\therefore x = \sqrt{1 + x}$$

On squaring both sides –

$$(x)^2 = (\sqrt{1 + x})^2$$

$$\rightarrow x^2 = 1 + x$$

$$\rightarrow x^2 - x - 1 = 0$$

$$\rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\rightarrow \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\rightarrow \frac{1 \pm \sqrt{1 + 4}}{2}$$

$$\rightarrow \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1 + \sqrt{5}}{2} \text{ OR } x = \frac{1 - \sqrt{5}}{2}$$

$$\therefore x = \frac{1 + \sqrt{5}}{2} \quad (x \text{ cannot have a negative value.})$$

Q: Considering another infinite surd $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}$ where the first term is $\sqrt{2 + \sqrt{2}}$, to repeat the entire process and to find the exact value for this surd.

ANSWER:

$$t_1 = \sqrt{2 + \sqrt{2}} = 1.8477...$$

$$t_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \quad \text{OR} \quad t_2 = \sqrt{2 + t_1} = 1.9615...$$

$$t_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \quad \text{OR} \quad t_3 = \sqrt{2 + t_2} = 1.9903...$$

As it is observed that $t_3 = \sqrt{2 + t_2}$, therefore it can be understood that $t_4 = \sqrt{2 + t_3}$ as the trend has been so until now.

Hence,

$$\rightarrow t_4 = \sqrt{2 + t_3} = 1.9975...$$

$$\rightarrow t_5 = \sqrt{2 + t_4} = 1.9993...$$

$$\rightarrow t_6 = \sqrt{2 + t_5} = 1.9998...$$

$$\rightarrow t_7 = \sqrt{2 + t_6} = 1.9999...$$

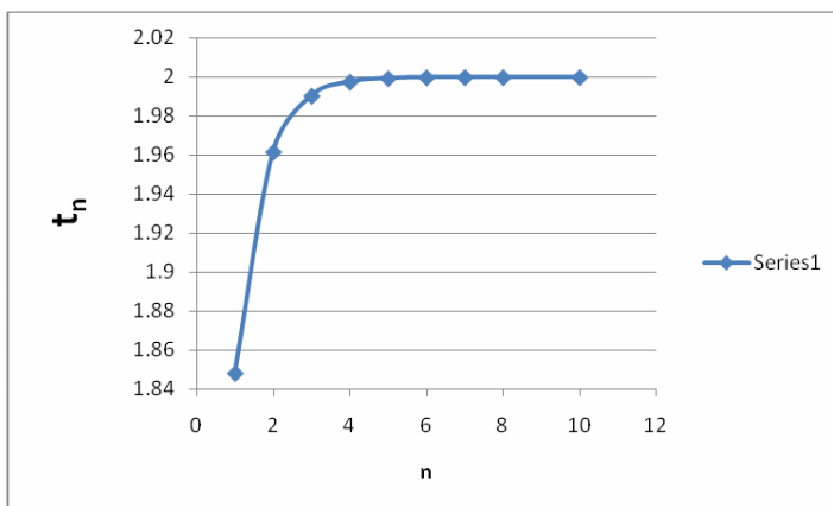
$$\rightarrow t_8 = \sqrt{2 + t_7} = 1.9999...$$

$$\rightarrow t_9 = \sqrt{2 + t_8} = 1.9999...$$

$$\rightarrow t_{10} = \sqrt{2 + t_9} = 1.9999...$$

[NOTE: VALUES OBTAINED USING THE GRAPHICAL DISPLAY CALCULATOR: CASIO fx- 9860G]

N	t_n
1	1.8477
2	1.9615
3	1.9903
4	1.9975
5	1.9993
6	1.9998
7	1.9999
8	1.9999
10	1.9999



From the above graph we realize as the value starts to increase after a certain number (1.999) it stabilizes and remains almost the same through out the graph. As we can see after 5 the value continues to remain the same till 10 and onwards. The general idea in both the question are that after the value starts to increase the graph stabilizes and the value remains the same.

Q: Find out the exact value for this surd ?

ANSWER:

$$\rightarrow t_{n+1} = \sqrt{2 + t_n}$$

As n becomes very large, $t_{n+1} - t_n = 0$ as n is ∞ .

$$\therefore t_{n+1} = t_n = x.$$

$$\therefore x = \sqrt{2 + x}$$

On squaring both sides –

$$\rightarrow (x)^2 = (\sqrt{2 + x})^2$$

$$\rightarrow x^2 = 2 + x$$

$$\rightarrow x^2 - x - 2 = 0$$

$$\rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$\rightarrow x = \frac{1 \pm \sqrt{1 + 8}}{2}$$

$$\rightarrow x = \frac{1 \pm \sqrt{9}}{2}$$

$$\rightarrow \therefore x = \frac{1 + 3}{2} \text{ OR } x = \frac{1 - 3}{2}$$

$$\therefore x = \frac{4}{2} = 2$$

Q: Now, considering the general infinite surd $\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}$ where the first term is $\sqrt{k + \sqrt{k}}$, find an expression for the exact value of this general infinite surd in terms of k .

ANSWER:

The formula in terms of $k \rightarrow c_{n+1} = \sqrt{k + c_n}$

The value of an infinite surd is not always an integer.

Q: Find some values of k that make the expression an integer. Find the general statement that represents all the values of k for which the expression is an integer.

ANSWER:

$$c_{n+1} = \sqrt{k + c_n}$$

$$c_{n+1} = c_n = x.$$

$$\therefore x = \sqrt{k + x}$$

On squaring both sides –

$$\rightarrow x^2 = k + x$$

$$\rightarrow x^2 - x - k = 0$$

$$\rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-k)}}{2(1)}$$

$$\rightarrow x = \frac{1 \pm \sqrt{1 + 4k}}{2}$$

$$\therefore x = \frac{1 + \sqrt{1 + 4k}}{2} \text{ OR } x = \frac{1 - \sqrt{1 + 4k}}{2}$$

$$\therefore x = \frac{1 + \sqrt{1 + 4k}}{2} \quad (x \text{ cannot be negative})$$

→ suppose:

$$(i) 1+4k=0$$

$$4k=0$$

$$\therefore k=0$$

$$(ii) 1+4k=9$$

$$4k=8$$

$$\therefore K=2$$

$$(iii) 1+4k=49$$

$$4k=48$$

$$\therefore k=12$$

Therefore, the sequence obtained is $-0, 2, 6, 12, \dots, n$

$$u_1 = 0$$

$$u_2 = 2$$

$$u_3 = 6 \rightarrow u_2 + 4 = 2+4$$

$$u_4 = 12 \rightarrow u_3 + 6 = 2+4+6$$

$$u_5 = 2+4+6+8$$

$$u_6 = 2+4+6+8+10+\dots$$

$$\therefore s_n = \frac{n}{2} \{2u_1 + (n-1)d\}$$

$$\rightarrow s_n = \frac{n}{2} \{2(2) + (n-1)2\}$$

$$\rightarrow s_n = \frac{n}{2} (4+2n-2)$$

$$\rightarrow s_n = \frac{n}{2} (2+2n)$$

$$\therefore s_n = \frac{2n}{2} + \frac{2n^2}{2}$$

$$\therefore u_n = n+n^2$$

Q: test the validity of your general statement using other values of k.

ANSWER:

(i) Suppose $n=5$

Then, on substituting 5 as the value of n in the general statement we get:-

$$s_n = 5 + 5^2 = 30$$

$$\therefore k = 30$$

Now substituting the value of k in the equation $x^2 - x - k = 0$ we get:

$$x^2 - x - 30 = 0$$

$$\rightarrow x^2 - 6x + 5x - 30 = 0$$

$$\rightarrow x(x-6) + 5(x-6) = 0$$

$$\rightarrow (x+5)(x-6) = 0$$

Thus, $x=6$ OR $x=-5$

Hence, proved.

(ii) Supposing $n=6$

Then, on substituting 6 as the value of n in the general statement we get:-

$$s_n = 6 + 6^2 = 42$$

$$\therefore k = 42$$

Now substituting the value of k in the equation $x^2 - x - k = 0$ we get:

$$x^2 - x - 42 = 0$$

$$\rightarrow x^2 - 7x + 6x - 42 = 0$$

$$\rightarrow x(x-7) + 6(x-7) = 0$$

$$\rightarrow (x+6)(x-7) = 0$$

Thus, $x=7$ OR $x=-6$

Hence, proved.

HENCE, IT CAN BE CONCLUDED THAT EVEN AFTER USING OTHER VALUES OF “k” THE ANSWER OBTAINED IS AN INTEGER WHICH PROVES THE OBTAINED GENERAL STATEMENT VALID.

The scope of the general statement is that the need to always obtain an integer value as the answer is always satisfied by this general statement. The limitation of the general statement is that the use of a negative integer is invalid.

CONCLUSION:

By this I conclude that them I learnt how to use the surds and the correct time to use them.