

The English School

IB Mathematics SL
Math Portfolio (Type1)

Infinite Summation

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Aim: The aim of this portfolio is to research about the summation of infinite series. In first place I will consider the general sequence with constant values for x and variables for a . The S_n , which is calculate the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$ will be calculated with the programme of Microsoft Excel and illustrated in graphs with Graphmatica; another computer based program, so that we can create a general statement which can be proven. Furthermore, to expand the investigation I will explore the same general sequence, but with variables in both a and x . I will accomplish this by doing the same method as the first exercises. Finally, I will conclude by showing the scopes/limitations of the general statement.

$$t_n = \frac{(x \ln a)^n}{n!}$$

Where,

$$t_0 = 1, t_1 = \frac{(x \ln a)}{1!}, t_2 = \frac{(x \ln a)^2}{2!}, t_3 = \frac{(x \ln a)^3}{3!}, \dots$$

We must take into account that factorial notation is in succession, this means that the factorial notation shows all the natural numbers from 1 to n , in such way that:

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

Following the first exercise, we will change the variables of terms x and a , in such way that $x = 1$ and $a = 2$. If we replace these values into the original sequence we obtain:

$$1, \frac{(\ln 2)}{1!}, \frac{(\ln 2)^2}{2!}, \frac{(\ln 2)^3}{3!}, \dots$$

In order to calculate the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$ we must know that the sum of the first 10 values is in progression. Hence, I will use Microsoft Excel in order to plot results in a suitable table. The first column will contain the different values of n , which come from 1 to 10. The second column will contain the results obtained by replacing each of the n values in the form of $\frac{(\ln 2)^n}{n!}$. And the third column will contain the gradual sum of each of the terms obtained in the second column. For example, the first value of the third column will be added to the second value of the second column giving the second value of the third column, and so on.

With the help of Excel the table will look like this:

n	Replacing in the form $\frac{(\ln 2)^n}{n!}$	S_n
0	1	1
1	0,69314718	1,69314718
2	0,24022651	1,93337369
3	0,05550411	1,9888778
4	0,00961813	1,99849593
5	0,00133336	1,99982928
6	0,00015404	1,99998332
7	1,5253E-05	1,99999857
8	1,3215E-06	1,99999989
9	1,0178E-07	1,99999999
10	7,0549E-09	2

Table 1: Shows the sum of the first 10 terms for $a = 2$

After analysing the previous results the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$ is 2.

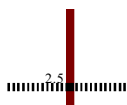
Likewise, we can represent the relationship between S_n and n in a graph by replacing the values of S_n in the y-axis and the values of n in the x-axis can do this. The graph should look like this:

n	S_n
0	1
1	1,69314718
2	1,93337369
3	1,9888778
4	1,99849593
5	1,99982928
6	1,99998332
7	1,99999857
8	1,99999989
9	1,99999999
10	2

Table 2: Shows the values of the graph of S_n against n of $a = 2$

After using the computer programme Graphmatica, the result of the graph is this:

Graph 1: Shows the relation between S_n and n in $\alpha = 2$



We can observe that the S_n values increase exponentially until they reach the number 2. At this point they stabilize. Hence, we can conclude that this sequence is convergent. In other words, it is approaching a definite limit as more of its terms are added. This happens because the S_n values are getting smaller rather than being bigger due to the factorial notation in the denominator. As the denominator gets bigger we obtain results with smaller numbers. From this information we can say that the S_n values tend to move to 2 if the variable is $\alpha = 2$, likewise the way n approaches to ∞ .

Next, we can consider the same sequence with $x = 1$, but changing the variable α to $\alpha = 3$. The result of the sequence is as following:

$$1, \frac{(\ln 3)}{1!}, \frac{(\ln 3)^2}{2!}, \frac{(\ln 3)^3}{3!}, \dots$$

Now that we know the sequence we can now get the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$. Like the sequence before, I will use Microsoft Excel in order to get the table and therefore get the sum of all terms.

The table is as following:

n	Replacing in the form $\frac{(\ln 3)^n}{n!}$	S_n
0	1	1
1	1,09861229	2,09861229
2	0,60347448	2,70208677
3	0,22099483	2,9230816
4	0,06069691	2,9837785
5	0,01333647	2,99711498
6	0,00244194	2,99955691
7	0,00038325	2,99994016
8	5,263E-05	2,99999279
9	6,4245E-06	2,99999922
10	7,058E-07	2,99999992

Table 3: Shows the sum of the first 10 terms for $\alpha = 3$

After analysing the previous results the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$ is 2,99999992.

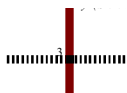
Likewise, we can represent the relationship between S_n and n in a graph by replacing the values of S_n in the y-axis and the values of n in the x-axis as seen in the previous sequence.

The table should look like this:

n	S_n
0	1
1	2,09861229
2	2,70208677
3	2,9230816
4	2,9837785
5	2,99711498
6	2,99955691
7	2,99994016
8	2,99999279
9	2,99999922
10	2,99999992

Table 4: Shows the values of the graph of S_n against n of $\alpha = 3$

After using the computer programme Graphmatica, the result of the graph is this:



Graph 2: Shows the relation between S_n and n in $\alpha = 3$

Same as the previous sequence, we can observe that the S_n values increase exponentially but they never reach 3. At this point they stabilize. Hence, we can conclude that this sequence is also convergent. This happens because the S_n values are getting smaller rather than being bigger due to the factorial notation in the denominator. As the denominator gets bigger we obtain results with smaller numbers. From this information we can say that the S_n values tend to move to 3 if the variable is $\alpha = 3$, likewise the way n approaches to ∞ .

Next, we will consider the same general sequence where $x = 1$ in such way that:

$$1, \frac{(\ln \alpha)}{1!}, \frac{(\ln \alpha)^2}{2!}, \frac{(\ln \alpha)^3}{3!}, \dots$$

In this sequence we will replace different values for variable α . In my case I will use 3 different values that can replace α : 4, 5 and 6. For each of them I will calculate the sum S_n of the first n terms of the above sequence for $0 \leq n \leq 10$. To do this I will use again the help of Microsoft Excel as done previously in other sequences.

n	Replacing in the form $\frac{(\ln a)^n}{n!}$	S_n
0	1	1
1	1,38629436	2,38629436
2	0,96090603	3,34720039
3	0,44403287	3,79123326
4	0,15389007	3,94512332
5	0,04266739	3,98779071
6	0,00985826	3,99764897
7	0,00195235	3,99960132
8	0,00033832	3,99993964
9	5,2112E-05	3,99999175
10	7,2242E-06	3,99999897

Table 5: The sum of S_n for $0 \leq n \leq 10$, given that $a = 4$

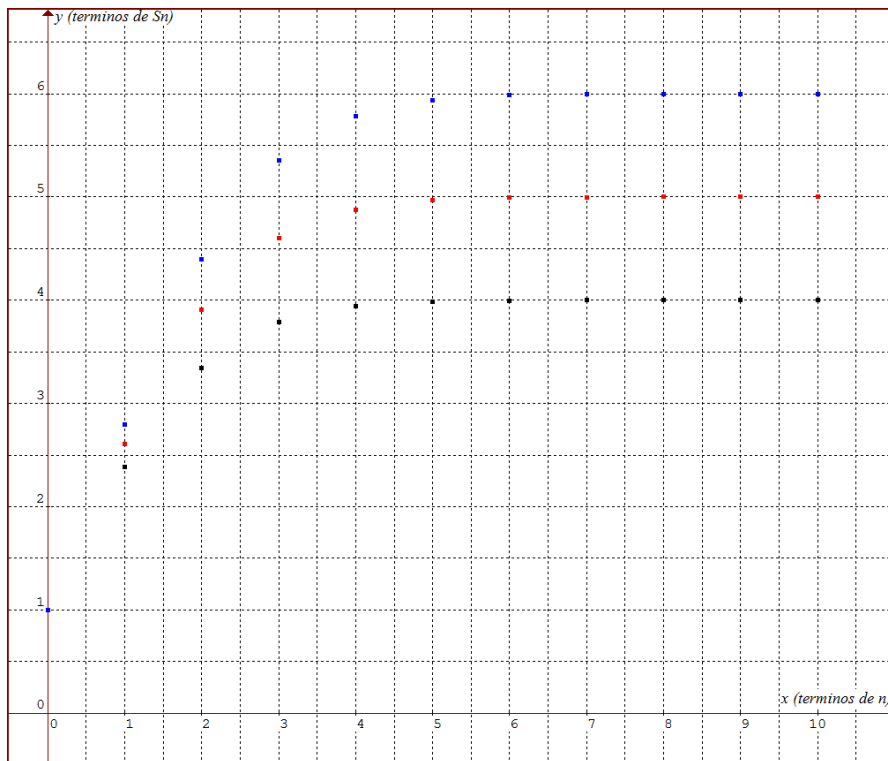
n	Replacing in the form $\frac{(\ln a)^n}{n!}$	S_n
0	1	1
1	1,60943791	2,60943791
2	1,2951452	3,90458311
3	0,69481859	4,5994017
4	0,27956685	4,87896855
5	0,0899891	4,96895765
6	0,02413864	4,99309629
7	0,00554995	4,99864624
8	0,00111654	4,99976278
9	0,00019967	4,99996244
10	3,2135E-05	4,99999458

Table 6: The sum of S_n for $0 \leq n \leq 10$, given that $a = 5$

n	Replacing in the form $\frac{(\ln a)^n}{n!}$	S_n
0	1	1
1	1,79175947	2,79175947
2	1,605201	4,39696047
3	0,95871136	5,35567183
4	0,42944504	5,78511687
5	0,15389244	5,93900931
6	0,04595637	5,98496569
7	0,01176325	5,99672894
8	0,00263461	5,99936356
9	0,00052451	5,99988807
10	9,398E-05	5,99998205

Table 7: The sum of S_n for $0 \leq n \leq 10$, given that $a = 6$

Analysing the results of each of the S_n values we can notice similar results to previous exercises. To make a deeper analysis of data I will plot the three tables on a single graph, showing only the results of the third column. Each of the final results will be represented by colour: $\alpha = 4$, $\alpha = 5$, $\alpha = 6$



Graph 3: Shows the relation between S_n and n , given $\alpha = 4, 5$ and 6

Looking in analysis, we can see that each of the replaced values of α increase exponentially to a certain point where they are stabilized. The point in which they stabilize is the replaced value of α and again, the reason for this is that all of the curves are convergent. So to generalize, we can confirm that the value of S_n tends to the replaced value of α in such way that the value of n tends to ∞ .

After looking previous investigations we are able to obtain a general statement that represents the infinite sum of the general sequence. The general statement is:

$$\sum_{n=0}^{\infty} \frac{(\ln \alpha)^n}{n!} \approx \alpha$$

This statement comes from all the results obtained in previous investigations due to the fact that every time that we replace the value of α in the sequence, including the result up to 10 and even the infinite, it will always give us the same value that was replaced in the progression. Even though, as the limits fluctuate it is said that it is approximately α , in other words it is precise but not accurate.

In order to proof the general statement I will do the same process as before but with a bigger number in such way that $a = 20$. So the first term of the second column of the table should look like this:

$$\sum_{n=0}^{\infty} \frac{(\ln 20)^n}{n!} \approx a$$

$$2.995732274 \approx a$$

Completing the table with the help of the spread sheet in Excel it should look like this:

0	1	1
1	2,99573227	3,99573227
2	4,48720593	8,4829382
3	4,48082254	12,9637607
4	3,35583617	16,3195969
5	2,01063735	18,3302343
6	1,00388853	19,3341228
7	0,4296259	19,7637487
8	0,16088052	19,9246292
9	0,05355055	19,9781798
10	0,01604231	19,9942221
11	0,00436895	19,998591
12	0,00109068	19,9996817
13	0,00025134	19,999933
14	5,3782E-05	19,9999868
15	1,0741E-05	19,9999868
16	2,0111E-06	19,9999976
17	3,5439E-07	19,9999996
18	5,8981E-08	20

Table 8: The sum of S_n for $0 \leq n \leq 18$, given that $a = 20$

As we can see from the general statement, the value that replaces a in the function will be result of the sum from 0 to ∞ . Also it is important to notice that the sequence reaches to number 20 meaning that from summation 18 to ∞ , the result will be 20.

Subsequent to expanding this investigation we will consider again our original sequence:

$$t_0 = 1, t_1 = \frac{(x \ln a)}{1!}, t_2 = \frac{(x \ln a)^2}{2!}, t_3 = \frac{(x \ln a)^3}{3!}, \dots$$

In this part of the exercise we will consider the expression $T_n(a, x)$, which represents the sum of the first n terms, for various values of a and x .

In first we will have the progression in which the variable $a = 2$, hence $T_p(2, x)$. To find out the sum of the first n terms we are going to calculate different values of x and calculate their results with Excel. The 4 different values of x are 0.01, 0.6, 3, and 5. The results of their summation are in the following tables:

0	1	1
1	0.006931472	1.00693147 2
2	2.40227E-05	1.00695549 4
3	5.55041E-08	1.00695555
4	9.61813E-11	1.00695555
5	1.33336E-13	1.00695555
6	1.54035E-16	1.00695555
7	1.52527E-19	1.00695555
8	1.32155E-22	1.00695555
9	1.01781E-25	1.00695555

Table 9: $T_p(2, 0.01)$

0	1	1
1	2.079441542	3.07944154 2
2	2.162038563	5.24148010 4
3	1.498610934	6.74009103 8
4	0.779068458	7.51915949 6
5	0.324005463	7.84316495 9
6	0.112291737	7.95545669 6
7	0.033357729	7.98881442 4
8	0.008670681	7.99748510 5
9	0.002003353	7.99948845 8

0	1	1
1	0.415888308	1.41588830 8
2	0.086481543	1.50236985 1
3	0.011988887	1.51435873 8
4	0.00124651	1.51560524 8
5	0.000103682	1.51570893
6	7.18667E-06	1.51571611 6
7	4.26979E-07	1.51571654 3
8	2.21969E-08	1.51571656 5
9	1.02572E-09	1.51571656 6

Table 10: $T_p(2, 0.6)$

0	1	1
1	3.465735903	4.46573590 3
2	6.005662674	10.4713985 8
3	6.938013583	17.4094121 6
4	6.011330692	23.4207428 5
5	4.166736921	27.5874797 7
6	2.406801624	29.9942814
7	1.191619828	31.1859012 3
8	0.516229953	31.7021311 8
9	0.198790742	31.9009219 2

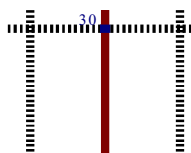
Table 11: $T_9(2, 3)$

Table 12: $T_9(2, 5)$

Based on the results from the tables and the sum until the term 9 we can now plot a graph that represents the relation with variable x . The graph will contain the following data

Table 13: Data of graph $T_9(2, x)$ vs x

x	$T_9(2, x)$
0.01	1.00695555
0.6	1.515716566
3	7.999488458



5	31.90092192
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Graph 3: Shows the relation between $T_9(2, x)$ vs x

Analysing the graph we can see that points increase exponentially as the curve fit line can show us. We can see that the values of $T_2(2, x)$ have increased by the expression $T_2 = x + 1$ from the original value of x . For example if $x = 3$, then the value of $T_2 = (3) + 1$, which is equal to 4.

The next exercise is that we will have the same progression, but this time we will change variable to $a = 3$, hence $T_3(3, x)$. To find out the sum of the first n terms we are going to calculate different values of x like we did before in Excel. The 4 different values of x are 1, 2, 4, and 8. The results of their summation are in the following tables:

0	1	1
1	1.098612289	2.098612289
2	0.60347448	2.702086769
3	0.220994827	2.923081596
4	0.060696908	2.983778504
5	0.013336474	2.997114978
6	0.002441936	2.999556913
7	0.000383249	2.999940162
8	5.26302E-05	2.999992792
9	6.42447E-06	2.999999217

Table 14: $T_3(3, 1)$

0	1	1
1	4.394449155	5.394449155
2	9.655591687	15.05004084
3	14.14366891	29.19370975
4	15.53840847	44.73211822
5	13.65654919	58.38866741
6	10.00216851	68.39083592
7	6.27914585	74.66998177
8	3.449173397	78.11915517
9	1.684135235	79.8032904

Table 15: $T_3(3, 4)$

0	1	1
1	2.197224577	3.197224577
2	2.413897922	5.611122499
3	1.767958614	7.379081112
4	0.971150529	8.350231642
5	0.426767162	8.776998804
6	0.156283883	8.933282687
7	0.049055827	8.982338514
8	0.013473334	8.995811848
9	0.003289327	8.999101174

Table 15: $T_3(3, 2)$

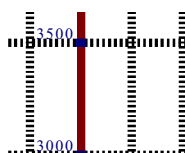
0	1	1
1	8.788898309	9.788898309
2	38.62236675	48.41126506
3	113.1493513	161.5606163
4	248.6145355	410.1751518
5	437.0095742	847.184726
6	640.1387846	1487.323511
7	803.7306688	2291.054179
8	882.9883895	3174.042569
9	862.2772404	4036.319809

Table 16: $T_3(3, 8)$

Based on the results from the tables and the sum until the term 9 we can now plot a graph that represents the relation with variable x like on the previous exercise. The graph will contain the following data:

Table 17: Data of graph $T_9(3, x)$ vs x

x	$T_9(3, x)$
1	2.999999217
2	8.999101174
4	79.8032904
8	4036.319809



Graph 4: Shows the relation between $T_9(3, x)$ vs x

Results on this graph are not surprising and actually they show the same values but with the variable $a = 3$. We can see that the values of $T_{15}(3, x)$ have increased by the expression $2x + 1$ from the original value of x . For example if $x = 2$, then the value of $T_{15} = 2(2) + 1$, which is equal to 5.

Based on the previous exercises we are now able to establish the general statement for $T_{15}(a, x)$ as n approaches ∞ . The general statement is as following

$$\sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} \approx a^x$$

This statement comes from all the results obtained in previous investigations due to the fact that every time that we replace the value of a and x in the sequence, including the result up to 10 and even the infinite, it will always give us the same value that was replaced in the progression, but this time it will be a^x . Even though, as the limits fluctuate it is said that it is approximately a^x , in other words it is precise but not accurate.

To test the validity of my general statement I will take other values of a and x .

I will consider the sequence with the expression of $T_{15}(4, 7)$. So the sequence will look like this. If it is correct the result will be 2401

$$t_0 = 1, t_1 = \frac{(7 \ln 4)}{1!}, t_2 = \frac{(7 \ln 4)^2}{2!}, t_3 = \frac{(7 \ln 4)^3}{3!}, \dots$$

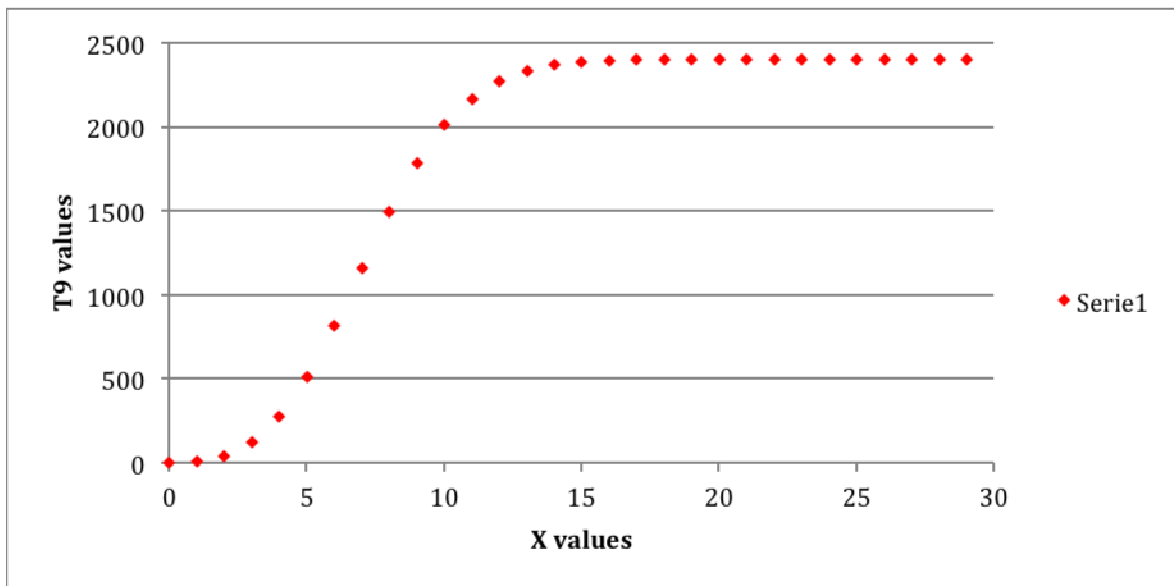
Hence the table with the summation will look like this:

Table 18: The sum of S_n for $0 \leq n \leq 29$ given that $a = 7$ and $x = 4$

Hence the table should look like this:

24	0.003941676	2400.998259
25	0.001227224	2400.999487
26	0.000367395	2400.999854
27	0.000105914	2400.99996
28	2.94426E-05	2400.999989
29	7.90245E-06	2400.999997

And therefore the graph should correspond to this:



Graph 5: Shows the relation between $T_{29}(4, 7)$ vs x

The greatest limitation that it has is the fact that you limit them of the sum not always are the same so the result is not always exact.

The general this general statement was obtained by means of the analysis of the tables and the graphic ones that were used throughout the exercise. When replacing by different values for a and x , the general proposal was fulfilled so that the S_n terms gave the result of a^x . In the first place for the x values when $a = 2$, and was fulfilled later for the values of x when values of $a = 3$, which demonstrated that the results could not be the same. After I did a deeper analysis of graphs and tables I was able to confirm the general statement by the importance of the relation that there is between the value of x that is replaced and the respective value for T_p .