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PORTFOLIO ASSIGNMENT TYPE 1

MATHEMATICAL INVESTIGATION

INFINITE SUMMATION

Set date: Monday, March 8th, 2010

Due date: Monday, March 15th, 2010

Name: Tra My Nguyen

Candidate number:

- *Technology used:*
    - *Microsoft word*
    - *Rapid-pi*
    - *Microsoft Excel*
    - *GDC (TI-84 Plus)*

In this portfolio, I will determine the general sequence t_n with different values of variables to find the formula to count the sum of the infinite sequence.

I will investigate the sum of infinite sequences t_n, where:

$$t_0 = 1,\ t_1 = \frac{x\ln a}{1},\ t_2 = \frac{(x\ln a)^2}{2\times 1},\ t_3 = \frac{(x\ln a)^3}{3\times 2\times 1},\ \ldots,\ t_n = \frac{(x\ln a)^n}{n!},\ \ldots$$

* a must be positive because according to the definition of logarithms:

$y = \log_x y \Rightarrow x>0,\ x \neq 1, y>0$

$\ln a = \log_e a \Rightarrow a>0$

(approximately e = 2.718281828)

* I will use the factorial notation:

$n! = n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times 3\times 2\times 1$

To see how the sum changes when a changes, firstly, I am going to consider the sequence above, where $x=1$ and $a=2$:

$$\Rightarrow \left\{1, \frac{\ln 2}{1!}, \frac{(\ln 2)^2}{2!}, \frac{(\ln 2)^3}{3!}, \ldots\right\}$$

Let's define S_n to be the sum of the first $(n+1)$ terms of the sequence, $0 \le n \le 10$. Using GDC, I will calculate the sums S_0, S_1, S_2, …, S_{10} (giving answers correct to 6 decimal places):

$S_0 = t_0 = 1$

$S_1 = S_0 + t_1 = 1 + \frac{\ln 2}{1!} = 1.693147$

$S_2 = S_1 + t_2 = 1.693147 + \frac{(\ln 2)^{2}}{2!} = 1.933373$

$S_3 = S_2 + t_3 = 1.933373 + \frac{(\ln 2)^{3}}{3!} = 1.988877$

$S_4 = S_3 + t_4 = 1.988877 + \frac{(\ln 2)^{4}}{4!} = 1.998495$

$S_5 = S_4 + t_5 = 1.998495 + \frac{(\ln 2)^{5}}{5!} = 1.999828$

$S_6 = S_5 + t_6 = 1.999828 + \frac{(\ln 2)^{6}}{6!} = 1.999982$

$S_7 = S_6 + t_7 = 1.999982 + \frac{(\ln 2)^{7}}{7!} = 1.999997$

$S_8 = S_7 + t_8 = 1.999997 + \frac{(\ln 2)^{8}}{8!} = 1.999998322$

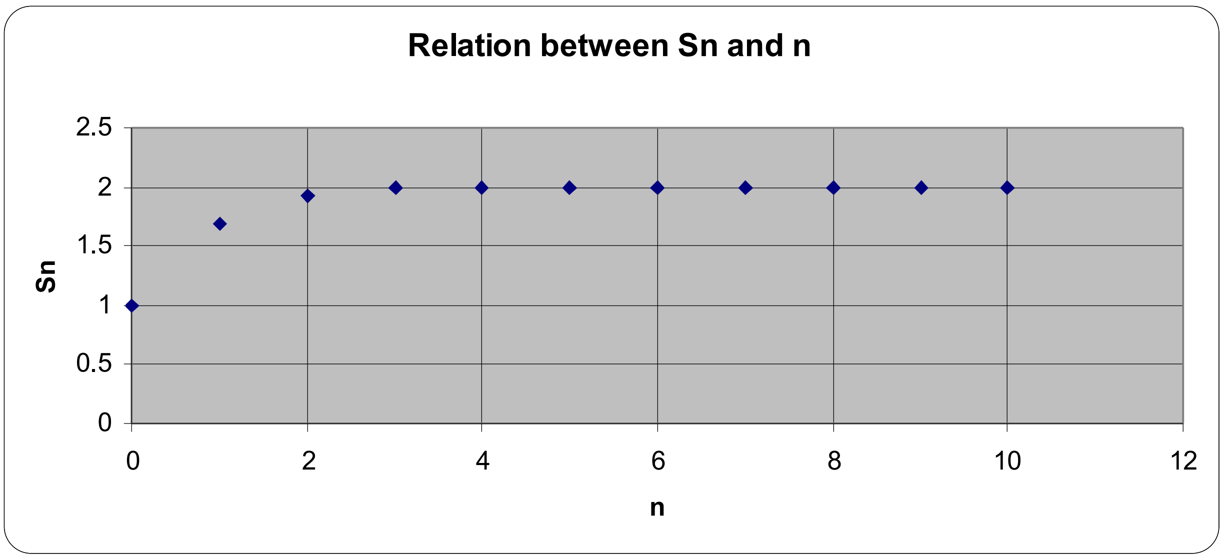
$S_9 = S_8 + t_9 = 1.999998322 + \frac{(\ln 2)^{9}}{9!} = 1.999998424$

$S_{10} = S_9 + t_{10} = 1.999998424 + \frac{(\ln 2)^{10}}{10!} = 1.999998431$

Now, using Microsoft Excel, I will plot the relation between S_n and n:

n	S_n
0	1
1	1.693147
2	1.933373
3	1.988877
4	1.998495
5	1.999828
6	1.999982

7	1.999997
8	1.999998
9	1.999998
10	1.999998



From this plot, I see that the values of S_n increase as values of n increase, but don't exceed 2, so the greatest value that S_n can have is 2. Therefore, it suggests about the values of S_n to be in domain $1 \le S_n \le 2$ as n approaches 1 when $x = 1$ and $a = 2$.

Now, doing similar as in first part, I am going to consider the sequence where $x = 1$ **and** $a = 3$ **:**

$$[\quad 1, \frac{\ln 3}{1!}, \frac{(\ln 3)^2}{2!}, \frac{(\ln 3)^3}{3!}, \ldots$$

Using GDC, I will calculate the sums $S_0, S_1, S_2, \ldots, S_{10}$:

$$S_0 = t_0 = 1$$

$$S_1 = S_0 + t_1 = 1 + \frac{\ln 3}{1!} = 2.098612$$

$$S_2 = S_1 + t_2 = 2.098612 + \frac{(\ln 3)^2}{2!} = 2.702087$$

$$S_3 = S_2 + t_3 = 2.702087 + \frac{\ln 2^{a_3}}{3!} = 2.923080$$

$$S_4 = S_3 + t_4 = 2.923080 + \frac{\ln 2^{a_4}}{4!} = 2.983777$$

$$S_5 = S_4 + t_5 = 2.983777 + \frac{\ln 2^{a_5}}{5!} = 2.997113$$

$$S_6 = S_5 + t_6 = 2.997113 + \frac{\ln 2^{a_6}}{6!} = 2.999555$$

$$S_7 = S_6 + t_7 = 2.999555 + \frac{\ln 2^{a_7}}{7!} = 2.999938$$

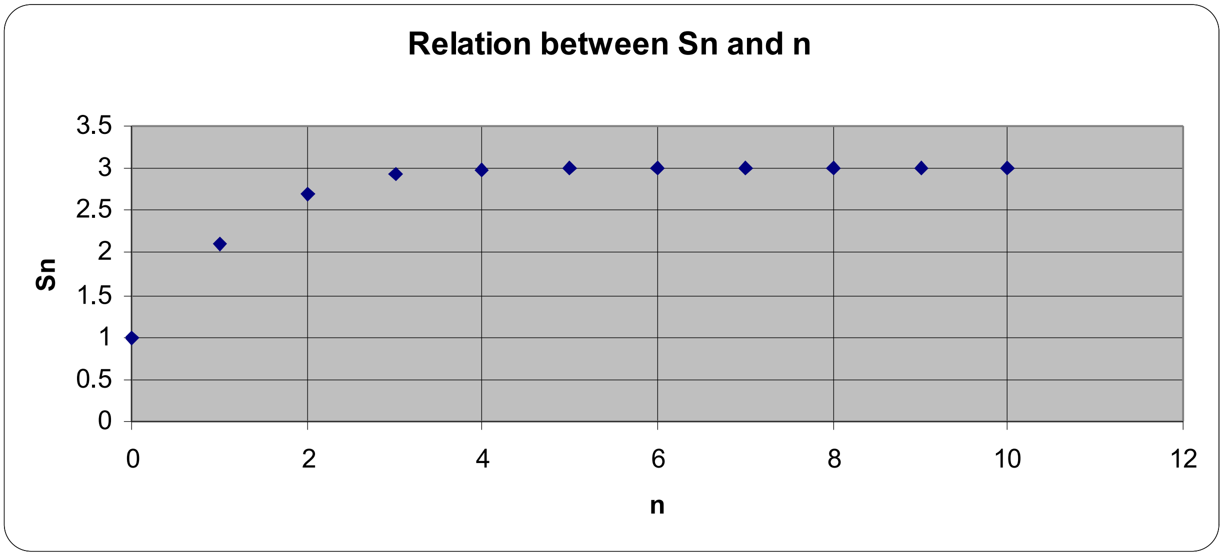
$$S_8 = S_7 + t_8 = 2.999938 + \frac{\ln 2^{a_8}}{8!} = 2.999991$$

$$S_9 = S_8 + t_9 = 2.999991 + \frac{\ln 2^{a_9}}{9!} = 2.999997$$

$$S_{10} = S_9 + t_{10} = 2.999997 + \frac{\ln 2^{a_{10}}}{10!} = 2.999998$$

Now, using Microsoft Excel, I will again plot the relation between S_n and n:

n	S_n
0	1
1	2.098612
2	2.702087
3	2.923080
4	2.983777
5	2.997113
6	2.999555
7	2.999938
8	2.999991
9	2.999997
10	2.999998



Again, I noticed that when $x = 1$ and $a = 3$, the values of S_n increase as values of n increase, but don't exceed *3*. So it suggests that S_n will be in domain $1 \le S_n \le 3$ as n approaches 1 .

Above, I have been supposing that the greatest value for the sum of infinite sequence S_n is *a*. And I want to check if it's correct with some different values of *a*.
Considering the general sequence where $x = 1$, I will calculate the sum S_n of the first *(n+1)* terms for $0 \le n \le 10$ for different values of a.
So, I will take random values for a, for example, $a = 7$ and $a = \pi$. And I will do exactly as above with $a = 2$ and $a = 3$ to see if there is any general statement for S_n.

Consider the sequence where $x = 1$ and $a = 7$:

$$\left[\ 1, \frac{\ln 7}{1!}, \frac{(\ln 7)^2}{2!}, \frac{(\ln 7)^3}{3!}, \ldots\right.$$

Using GDC, I will calculate the sums $S_0, S_1, S_2, \ldots, S_{10}$:

$$S_0 = t_0 = 1$$

$$S_1 = S_0 + t_1 = 1 + \frac{\ln 7}{1!} = 2.945910$$

$$S_2 = S_1 + t_2 = 2.945910 + \frac{(\ln 7)^2}{2!} = 4.839193$$

$$S_3 = S_2 + t_3 = 4.839193 + \frac{(\ln 7)^{3}}{3!} = 6.067246$$

$$S_4 = S_3 + t_4 = 6.067246 + \frac{(\ln 7)^{4}}{4!} = 6.664666$$

$$S_5 = S_4 + t_5 = 6.664666 + \frac{(\ln 7)^{5}}{5!} = 6.897171$$

$$S_6 = S_5 + t_6 = 6.897171 + \frac{(\ln 7)^{6}}{6!} = 6.972577$$

$$S_7 = S_6 + t_7 = 6.972577 + \frac{(\ln 7)^{7}}{7!} = 6.993539$$

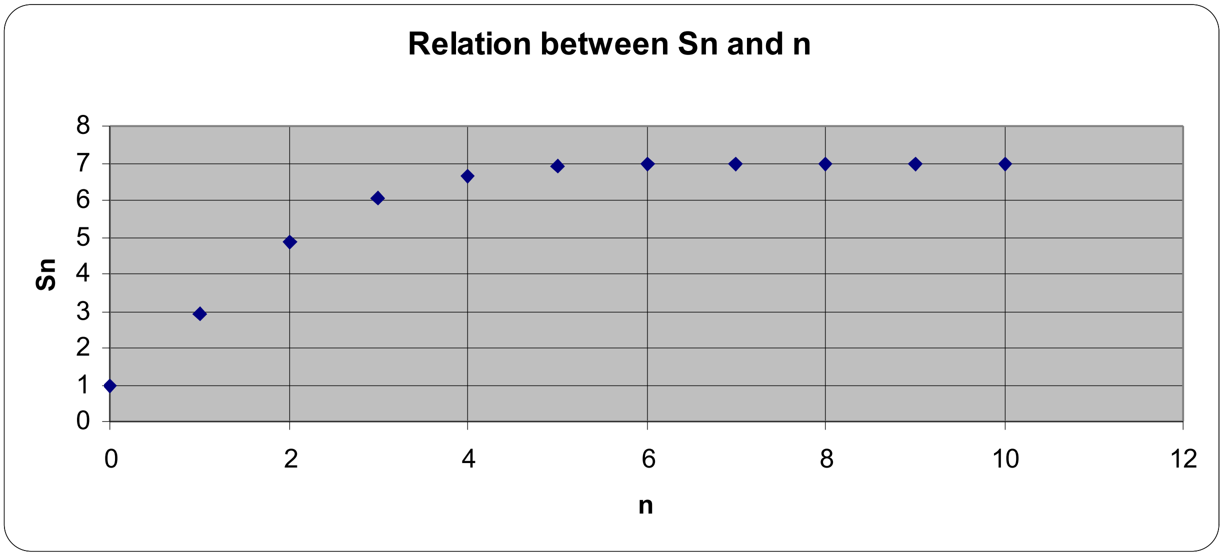
$$S_8 = S_7 + t_8 = 6.993539 + \frac{(\ln 7)^{8}}{8!} = 6.998638$$

$$S_9 = S_8 + t_9 = 6.998638 + \frac{(\ln 7)^{9}}{9!} = 6.999740$$

$$S_{10} = S_9 + t_{10} = 6.999740 + \frac{(\ln 7)^{10}}{10!} = 6.999956$$

Using Microsoft Excel, I plot the relation between S_n and n:

n	S_n
0	1
1	2.945910
2	4.839193
3	6.067246
4	6.664666
5	6.897171
6	6.972577
7	6.993539
8	6.998638
9	6.999740
10	6.999956



And I noticed that S_n increases when n increases, and values of S_n seem like not to exceed *7*. So, the domain for infinite sum of the general sequence where $x = 1$ and $a = 7$ is suggested to be $1 \leq S_n \leq 7$.

Now, moving on to the second example, I will consider the sequence where $x = 1$ and $a = \pi$:

$$\left[\ 1, \frac{\ln a}{1!}, \frac{(\ln a)^2}{2!}, \frac{(\ln a)^3}{3!}, \ldots\right.$$

So using GDC, I calculate the sums S_0, S_1, S_2, …, S_{10}:

$$S_0 = t_0 = 1$$

$$S_1 = S_0 + t_1 = 1 + \frac{\ln a}{1!} = 2.144730$$

$$S_2 = S_1 + t_2 = 2.144730 + \frac{(\ln a)^2}{2!} = 2.799933$$

$$S_3 = S_2 + t_3 = 2.799933 + \frac{(\ln a)^3}{3!} = 3.049943$$

$$S_4 = S_3 + t_4 = 3.049943 + \frac{(\ln a)^4}{4!} = 3.121492$$

$$S_5 = S_4 + t_5 = 3.121492 + \frac{(\ln a)^5}{5!} = 3.137873$$

$$S_6 = S_5 + t_6 = 3.137873 + \frac{(\ln a)^6}{6!} = 3.140998$$

$$S_7 = S_6 + t_7 = 3.140998 + \frac{a_7}{7!} = 3.141509$$

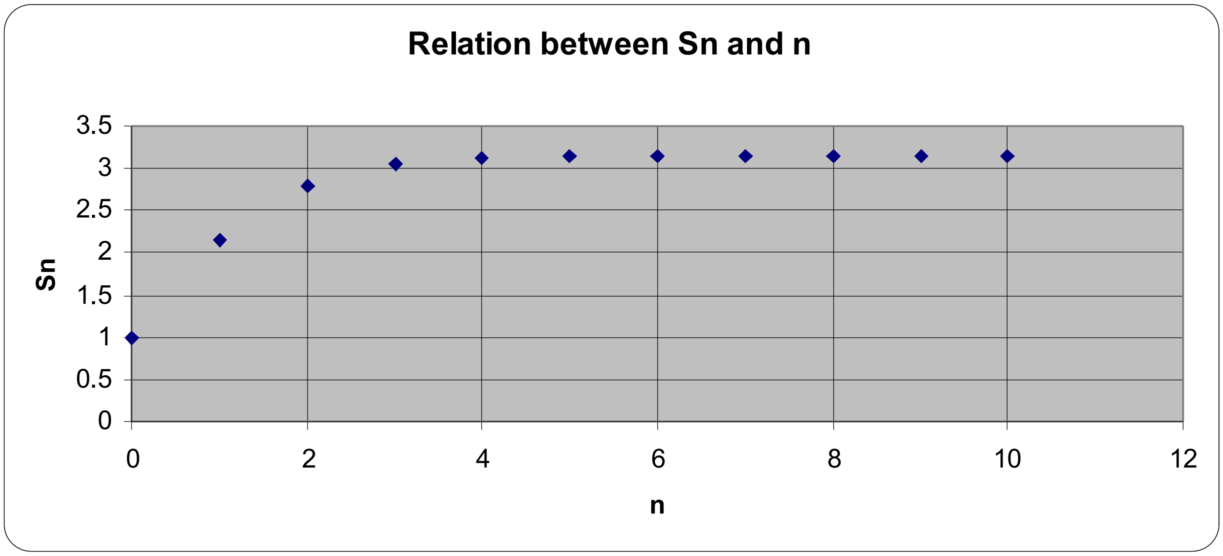
$$S_8 = S_7 + t_8 = 3.141509 + \frac{a_8}{8!} = 3.141582$$

$$S_9 = S_8 + t_9 = 3.141582 + \frac{a_9}{9!} = 3.141591$$

$$S_{10} = S_9 + t_{10} = 3.131591 + \frac{a_{10}}{10!} = 3.141592$$

Then, using Microsoft Excel, I plot the relation between S_n and n:

n	S_n
0	1
1	2.144730
2	2.799933
3	3.049943
4	3.121492
5	3.137873
6	3.140998
7	3.141509
8	3.141582
9	3.141591
10	3.141592



Knowing that $\pi = 3.141593$ (correct to six decimal places), I noticed that in the sequence given where $x = 1$ and $a = \pi$, S_n increases as n increases, and doesn't exceed π. So domain for the infinite sum S_n here is again suggested to be $1 \leq S_n \leq \pi$.

<u>Now let's analyse the initial general sequence</u>:

$t_0 = 1,\ t_1 = \frac{x \ln a}{1},\ t_2 = \frac{(x \ln a)^2}{2 \times 1},\ t_3 = \frac{(x \ln a)^3}{3 \times 2 \times 1},\ \ldots,\ t_n = \frac{(x \ln a)^n}{n!},\ \ldots.$

If I substitute $(x \ln a)$ with m, I can have a sequence like this:

$t_0 = 1,\ t_1 = m,\ t_2 = \frac{m^2}{2!},\ t_3 = \frac{m^3}{3!},\ \ldots$

And the sum of these infinite terms is:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

On the other hand, as defined by **power series expansion**, we have:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Therefore, we see that the infinite sum can be counted by $\boldsymbol{e^m}$, where $m = xlna$.

So what I noticed from here is (hypothesis):

- *Values of S_n increase as values of n increase.*
- *The greatest value for S_n infinite is suggested to be a.* [$1 \leq S_n \leq a$.
- *The statement to find the sum of infinite sequence is suggested to be* $\boldsymbol{e^{(xlna)}}$.

Now, it would be very interesting to expand this investigation to determine the sum of the infinite sequence t_n, where:

$t_0 = 1,\ t_1 = \dfrac{xlna}{1},\ t_2 = \dfrac{(xlna)^2}{2 \times 1},\ t_3 = \dfrac{(xlna)^3}{3 \times 2 \times 1},\ \ldots,\ t_n = \dfrac{(xlna)^n}{n!},\ \ldots$

T_n (a,x) *is defined to be the sum of the first n terms, for variable values of a and x.*

E.g.: T_6 (2,3) is the sum of the first 6 terms when a = 2 and x = 3.

*Let **a = 2**. I will calculate T_9 (2,x) for various positive values of x, for example: 1,2, 3, 4, 5, 6, 7, 9,10,11.*

Firstly, using GDC, I will calculate the sum of the first 9 terms when **a = 2 and x = 1**:

$$T_9(2,1) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8$$

$$= 1 + \frac{\ln 2}{1!} + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} + \frac{(\ln 2)^5}{5!} + \frac{(\ln 2)^6}{6!} + \frac{(\ln 2)^7}{7!} + \frac{(\ln 2)^8}{8!}$$

$$= 1.999998322$$

When **a = 2 and x = 2**:

$T_9(2,2) =$

$$1 + \frac{2\ln 2}{1!} + \frac{(2\ln 2)^2}{2!} + \frac{(2\ln 2)^3}{3!} + \frac{(2\ln 2)^4}{4!} + \frac{(2\ln 2)^5}{5!} + \frac{(2\ln 2)^6}{6!} + \frac{(2\ln 2)^7}{7!} + \frac{(2\ln 2)^8}{8!}$$

$$= 3.999983$$

When **a = 2 and x = 3**:

$T_9(2,3) =$

$$1 + \frac{3\ln 2}{1!} + \frac{(3\ln 2)^2}{2!} + \frac{(3\ln 2)^3}{3!} + \frac{(3\ln 2)^4}{4!} + \frac{(3\ln 2)^5}{5!} + \frac{(3\ln 2)^6}{6!} + \frac{(3\ln 2)^7}{7!} + \frac{(3\ln 2)^8}{8!}$$

$$= 7.997486$$

When **a = 2 and x = 4**:

$T_9\,(2,4) =$

$$1 + \frac{4\ln 2}{1!} + \frac{(4\ln 2)^{2}}{2!} + \frac{(4\ln 2)^{3}}{3!} + \frac{(4\ln 2)^{4}}{4!} + \frac{(4\ln 2)^{5}}{5!} + \frac{(4\ln 2)^{6}}{6!} + \frac{(4\ln 2)^{7}}{7!} + \frac{(4\ln 2)^{8}}{8!}$$

$= 15.963512$

When **a = 2 and x = 5**:

$T_9\,(2,5) =$

$$1 + \frac{5\ln 2}{1!} + \frac{(5\ln 2)^{2}}{2!} + \frac{(5\ln 2)^{3}}{3!} + \frac{(5\ln 2)^{4}}{4!} + \frac{(5\ln 2)^{5}}{5!} + \frac{(5\ln 2)^{6}}{6!} + \frac{(5\ln 2)^{7}}{7!} + \frac{(5\ln 2)^{8}}{8!}$$

$= 31.702131$

When **a = 2 and x = 6**:

$T_9\,(2,6) =$

$$1 + \frac{6\ln 2}{1!} + \frac{(6\ln 2)^{2}}{2!} + \frac{(6\ln 2)^{3}}{3!} + \frac{(6\ln 2)^{4}}{4!} + \frac{(6\ln 2)^{5}}{5!} + \frac{(6\ln 2)^{6}}{6!} + \frac{(6\ln 2)^{7}}{7!} + \frac{(6\ln 2)^{8}}{8!}$$

$= 62.305296$

When **a = 2 and x = 7**:

$T_9\,(2,7) =$

$$1 + \frac{7\ln 2}{1!} + \frac{(7\ln 2)^{2}}{2!} + \frac{(7\ln 2)^{3}}{3!} + \frac{(7\ln 2)^{4}}{4!} + \frac{(7\ln 2)^{5}}{5!} + \frac{(7\ln 2)^{6}}{6!} + \frac{(7\ln 2)^{7}}{7!} + \frac{(7\ln 2)^{8}}{8!}$$

$= 120.465723$

When **a = 2 and x = 9**:

$T_9\,(2,9) =$

$$1 + \frac{9\ln 2}{1!} + \frac{(9\ln 2)^{2}}{2!} + \frac{(9\ln 2)^{3}}{3!} + \frac{(9\ln 2)^{4}}{4!} + \frac{(9\ln 2)^{5}}{5!} + \frac{(9\ln 2)^{6}}{6!} + \frac{(9\ln 2)^{7}}{7!} + \frac{(9\ln 2)^{8}}{8!}$$

$= 420.699406$

When **a = 2 and x = 10**:

$T_9\,(2,10) =$

$$1 + \frac{10\ln 2}{1!} + \frac{(10\ln 2)^{2}}{2!} + \frac{(10\ln 2)^{3}}{3!} + \frac{(10\ln 2)^{4}}{4!} + \frac{(10\ln 2)^{5}}{5!} + \frac{(10\ln 2)^{6}}{6!} + \frac{(10\ln 2)^{7}}{7!} + \frac{(10\ln 2)^{8}}{8!}$$

$= 755.692615$

When **a = 2 and x = 11**:

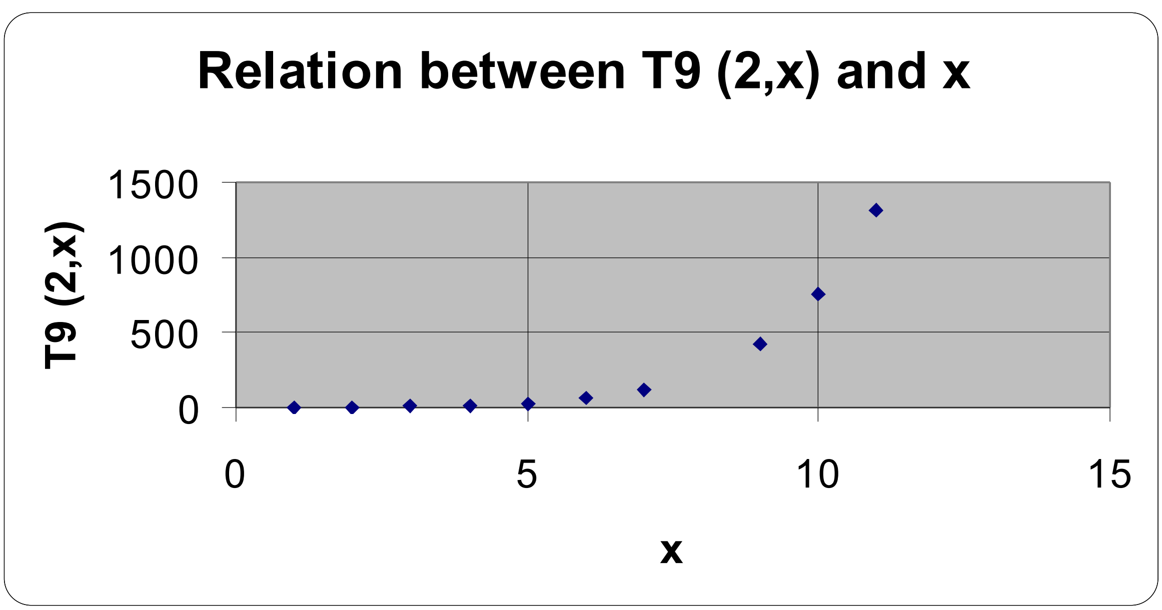
T_9 (2,11) =

$$1 + \frac{11\ln 2}{1!} + \frac{(11\ln 2)^2}{2!} + \frac{(11\ln 2)^3}{3!} + \frac{(11\ln 2)^4}{4!} + \frac{(11\ln 2)^5}{5!} + \frac{(11\ln 2)^6}{6!} + \frac{(11\ln 2)^7}{7!} + \frac{(11\ln 2)^8}{8!}$$

= 1320.526575

Using Microsoft Excel, I will plot the relation between T_9 (2,x) and x:

x	T_9 (2,x)
1	1.999998
2	3.999983
3	7.997486
4	15.963512
5	31.702131
6	62.305296
7	120.465723
9	420.699406
10	755.692615
11	1320.526575



From this plot, I can observe that as x increases, T_9 (2,x) is suggested to increase.

*Now, let **a = 3**. I will calculate the sum $T_9\ (3,x)$ for various positive values of x, for example: 1, 2, 3, 4, 5, 6, 7, 8, 9, 12.*

Using GDC, I am going to calculate the sum of the first 9 terms when ***a = 3* and *x = 1***:

$$T_9\ (3,1) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8$$

$$= 1 + \frac{\ln 3}{1!} + \frac{(\ln 3)^2}{2!} + \frac{(\ln 3)^3}{3!} + \frac{(\ln 3)^4}{4!} + \frac{(\ln 3)^5}{5!} + \frac{(\ln 3)^6}{6!} + \frac{(\ln 3)^7}{7!} + \frac{(\ln 3)^8}{8!}$$

$$= 2.999991$$

When ***a = 3* and *x = 2***:

$T_9\ (3,2) =$

$$1 + \frac{2\ln 3}{1!} + \frac{(2\ln 3)^2}{2!} + \frac{(2\ln 3)^3}{3!} + \frac{(2\ln 3)^4}{4!} + \frac{(2\ln 3)^5}{5!} + \frac{(2\ln 3)^6}{6!} + \frac{(2\ln 3)^7}{7!} + \frac{(2\ln 3)^8}{8!}$$

$$= 8.995813$$

When ***a = 3* and *x = 3***:

$T_9\ (3,3) =$

$$1 + \frac{3\ln 3}{1!} + \frac{(3\ln 3)^2}{2!} + \frac{(3\ln 3)^3}{3!} + \frac{(3\ln 3)^4}{4!} + \frac{(3\ln 3)^5}{5!} + \frac{(3\ln 3)^6}{6!} + \frac{(3\ln 3)^7}{7!} + \frac{(3\ln 3)^8}{8!}$$

$$= 26.814822$$

When ***a = 3* and *x = 4***:

$T_9\ (3,4) =$

$$1 + \frac{4\ln 3}{1!} + \frac{(4\ln 3)^2}{2!} + \frac{(4\ln 3)^3}{3!} + \frac{(4\ln 3)^4}{4!} + \frac{(4\ln 3)^5}{5!} + \frac{(4\ln 3)^6}{6!} + \frac{(4\ln 3)^7}{7!} + \frac{(4\ln 3)^8}{8!}$$

$$= 78.119155$$

When ***a = 3* and *x = 5***:

$T_9\ (3,5) =$

$$1 + \frac{5\ln 3}{1!} + \frac{(5\ln 3)^2}{2!} + \frac{(5\ln 3)^3}{3!} + \frac{(5\ln 3)^4}{4!} + \frac{(5\ln 3)^5}{5!} + \frac{(5\ln 3)^6}{6!} + \frac{(5\ln 3)^7}{7!} + \frac{(5\ln 3)^8}{8!}$$

$$= 217.471547$$

When ***a = 3* and *x = 6***:

$T_9\ (3,6) =$

$$1 + \frac{6\ln 3}{1!} + \frac{(6\ln 3)^2}{2!} + \frac{(6\ln 3)^3}{3!} + \frac{(6\ln 3)^4}{4!} + \frac{(6\ln 3)^5}{5!} + \frac{(6\ln 3)^6}{6!} + \frac{(6\ln 3)^7}{7!} + \frac{(6\ln 3)^8}{8!}$$

= 569.033838

When ***a* = 3 and *x* = 7**:

$T_9\,(3,7) =$

$$1 + \frac{7\ln 3}{1!} + \frac{(7\ln 3)^2}{2!} + \frac{(7\ln 3)^3}{3!} + \frac{(7\ln 3)^4}{4!} + \frac{(7\ln 3)^5}{5!} + \frac{(7\ln 3)^6}{6!} + \frac{(7\ln 3)^7}{7!} + \frac{(7\ln 3)^8}{8!}$$

= 1390.256866

When ***a* = 3 and *x* = 8**:

$T_9\,(3,8) =$

$$1 + \frac{8\ln 3}{1!} + \frac{(8\ln 3)^2}{2!} + \frac{(8\ln 3)^3}{3!} + \frac{(8\ln 3)^4}{4!} + \frac{(8\ln 3)^5}{5!} + \frac{(8\ln 3)^6}{6!} + \frac{(8\ln 3)^7}{7!} + \frac{(8\ln 3)^8}{8!}$$

= 3174.042570

When ***a* = 3 and *x* = 9**:

$T_9\,(3,9) =$

$$1 + \frac{9\ln 3}{1!} + \frac{(9\ln 3)^2}{2!} + \frac{(9\ln 3)^3}{3!} + \frac{(9\ln 3)^4}{4!} + \frac{(9\ln 3)^5}{5!} + \frac{(9\ln 3)^6}{6!} + \frac{(9\ln 3)^7}{7!} + \frac{(9\ln 3)^8}{8!}$$

= 6802.981104

When ***a* = 3 and *x* = 12**:

$T_9\,(3,12) =$

$$1 + \frac{12\ln 3}{1!} + \frac{(12\ln 3)^2}{2!} + \frac{(12\ln 3)^3}{3!} + \frac{(12\ln 3)^4}{4!} + \frac{(12\ln 3)^5}{5!} + \frac{(12\ln 3)^6}{6!} + \frac{(12\ln 3)^7}{7!}$$

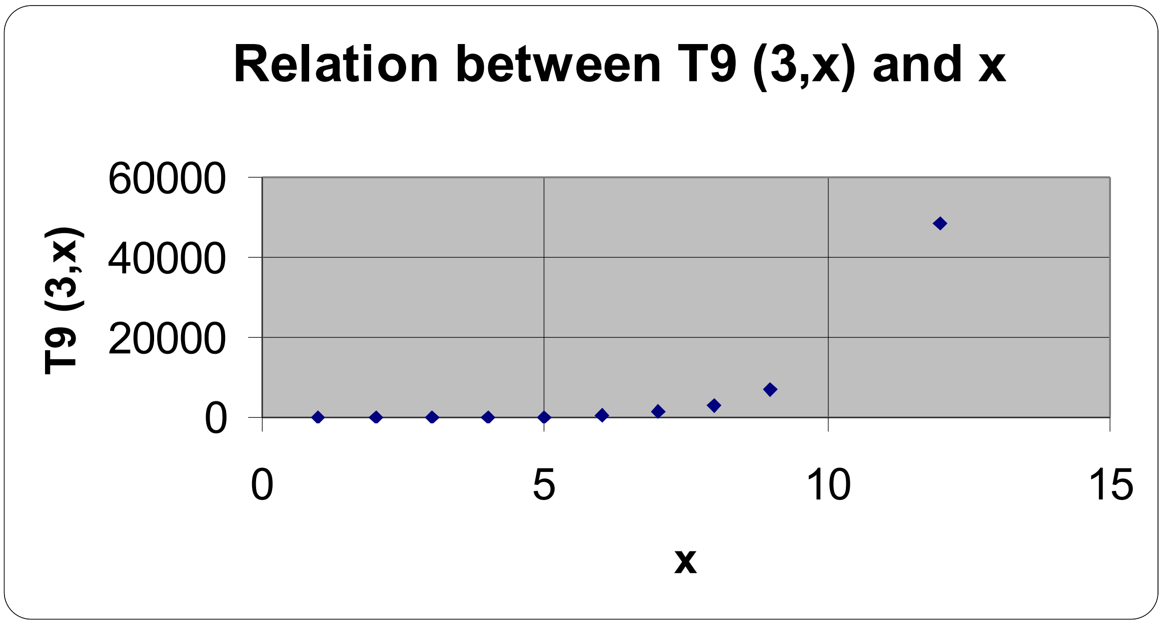
$$+ \frac{(12\ln 3)^8}{8!}$$

= 48714.21475

Now, using Microsoft Excel, I plot the relation between $T_9\,(3,x)$ and x:

x	T_9 (3,x)
1	2.999991
2	8.995813
3	26.814822
4	78.119155

5	217.471547
6	569.033838
7	1390.256866
8	3174.042570
9	6802.981104
12	48714.21475



Here, I also see that values of $T_9\ (3,x)$ increase as values of x increase, $T_9(3,x)$ seems to increase.

To find the general statement for T_n (a,x) as n approaches 1 , I will continue with this analysis until realising appropriate formula.

So let's take another value for a and see if we get the same notation as above.

For example, let $\boldsymbol{a} = \pi$, using GDC, I will calculate $T_7\ \pi,x$ for variable positive values of x. Let's take 1, 2, 3, 4, 5, 6, 7, 9, 11, 13:

When $\boldsymbol{a} = \pi$ **and** $\boldsymbol{x = 1}$:

$$T_7\,(\pi,1) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6$$

$$= 1 + \frac{\text{[illegible]}}{1!} + \frac{\text{[illegible]}^{2}}{2!} + \frac{\text{[illegible]}^{3}}{3!} + \frac{\text{[illegible]}^{4}}{4!} + \frac{\text{[illegible]}^{5}}{5!} + \frac{\text{[illegible]}^{6}}{6!}$$

$= 3.140998$

When ***a*** **=** π **and** ***x = 2***:

$$T_7(\pi,2) = 1 + \frac{2\ln\pi}{1!} + \frac{(2\ln\pi)^2}{2!} + \frac{(2\ln\pi)^3}{3!} + \frac{(2\ln\pi)^4}{4!} + \frac{(2\ln\pi)^5}{5!} + \frac{(2\ln\pi)^6}{6!}$$

$= 11.579476$

When ***a*** **=** π **and** ***x = 3***:

$$T_7(\pi,3) = 1 + \frac{3\ln\pi}{1!} + \frac{(3\ln\pi)^2}{2!} + \frac{(3\ln\pi)^3}{3!} + \frac{(3\ln\pi)^4}{4!} + \frac{(3\ln\pi)^5}{5!} + \frac{(3\ln\pi)^6}{6!}$$

$= 29.135563$

When ***a*** **=** π **and** ***x = 4***:

$$T_7(\pi,4) = 1 + \frac{4\ln\pi}{1!} + \frac{(4\ln\pi)^2}{2!} + \frac{(4\ln\pi)^3}{3!} + \frac{(4\ln\pi)^4}{4!} + \frac{(4\ln\pi)^5}{5!} + \frac{(4\ln\pi)^6}{6!}$$

$= 79.954196$

When ***a*** **=** π **and** ***x = 5***:

$$T_7(\pi,5) = 1 + \frac{5\ln\pi}{1!} + \frac{(5\ln\pi)^2}{2!} + \frac{(5\ln\pi)^3}{3!} + \frac{(5\ln\pi)^4}{4!} + \frac{(5\ln\pi)^5}{5!} + \frac{(5\ln\pi)^6}{6!}$$

$= 199.094837$

When ***a*** **=** π **and** ***x = 6***:

$$T_7(\pi,6) = 1 + \frac{6\ln\pi}{1!} + \frac{(6\ln\pi)^2}{2!} + \frac{(6\ln\pi)^3}{3!} + \frac{(6\ln\pi)^4}{4!} + \frac{(6\ln\pi)^5}{5!} + \frac{(6\ln\pi)^6}{6!}$$

$= 451.373522$

When ***a*** **=** π **and** ***x = 7***:

$$T_7(\pi,7) = 1 + \frac{7\ln\pi}{1!} + \frac{(7\ln\pi)^2}{2!} + \frac{(7\ln\pi)^3}{3!} + \frac{(7\ln\pi)^4}{4!} + \frac{(7\ln\pi)^5}{5!} + \frac{(7\ln\pi)^6}{6!}$$

$= 941.654242$

When ***a*** **=** π **and** ***x = 9***:

$$T_7(\pi,9) = 1 + \frac{9\ln\pi}{1!} + \frac{(9\ln\pi)^2}{2!} + \frac{(9\ln\pi)^3}{3!} + \frac{(9\ln\pi)^4}{4!} + \frac{(9\ln\pi)^5}{5!} + \frac{(9\ln\pi)^6}{6!}$$

$= 3344.217931$

When ***a*** **=** π **and** ***x = 11***:

$$T_7(\pi,11) = 1 + \frac{11\ln\pi}{1!} + \frac{(11\ln\pi)^2}{2!} + \frac{(11\ln\pi)^3}{3!} + \frac{(11\ln\pi)^4}{4!} + \frac{(11\ln\pi)^5}{5!} + \frac{(11\ln\pi)^6}{6!}$$

$= 9647.896312$

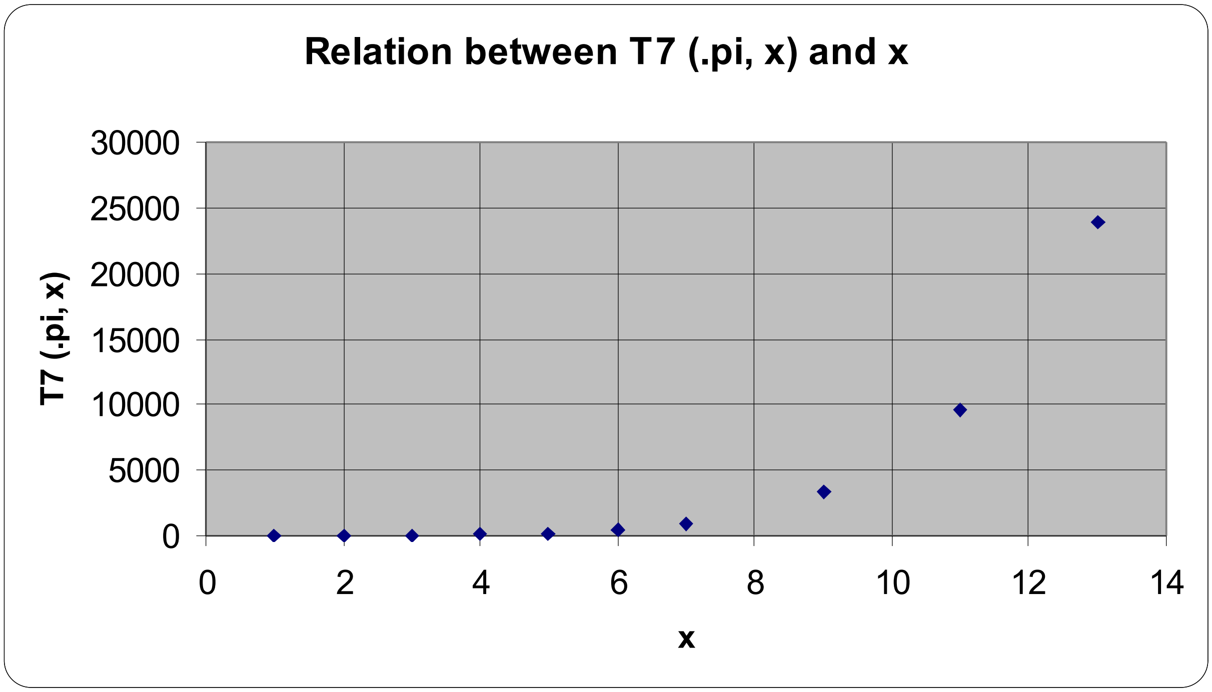
When $\boldsymbol{a = \pi}$ **and** $\boldsymbol{x = 13}$:

$$T_7(\pi,13) = 1 + \frac{13\ln\pi}{1!} + \frac{(13\ln\pi)^2}{2!} + \frac{(13\ln\pi)^3}{3!} + \frac{(13\ln\pi)^4}{4!} + \frac{(13\ln\pi)^5}{5!} + \frac{(13\ln\pi)^6}{6!}$$

$= 23886.45538$

And now, using Microsoft Excel, I will plot the relation between $T_7(\pi,x)$ and x:

x	$T_7(\pi,x)$
1	3.140998
2	11.579476
3	29.135563
4	79.954196
5	199.094837
6	451.373522
7	941.654242
9	3344.217931
11	9647.896312
13	23886.45538



From the plot, I observed that $T_7\ (\pi,x)$ increases as x increases.

<u>What I want to find out is</u>: How does $T_n\ (a,x)$ increase as n approaches 1 ?

So let's test the validity of the statement found above (e^{xlna}) by counting the infinite sum for variable a and x.

When $\boldsymbol{x} = \boldsymbol{1}$ and $\boldsymbol{a} = \boldsymbol{2}$: $S_n = e^{ln2} = 2 = \boldsymbol{2^1}$
When $\boldsymbol{x} = \boldsymbol{2}$ and $\boldsymbol{a} = \boldsymbol{2}$: $S_n = e^{2ln2} = 4 = \boldsymbol{2^2}$
When $\boldsymbol{x} = \boldsymbol{3}$ and $\boldsymbol{a} = \boldsymbol{2}$: $S_n = e^{3ln2} = 8 = \boldsymbol{2^3}$
When $\boldsymbol{x} = \boldsymbol{4}$ and $\boldsymbol{a} = \boldsymbol{2}$: $S_n = e^{4ln2} = 16 = \boldsymbol{2^4}$

When $\boldsymbol{x} = \boldsymbol{1}$ and $\boldsymbol{a} = \boldsymbol{5}$: $S_n = e^{ln5} = 5 = \boldsymbol{5^1}$
When $\boldsymbol{x} = \boldsymbol{2}$ and $\boldsymbol{a} = \boldsymbol{5}$: $S_n = e^{2ln5} = 25 = \boldsymbol{5^2}$
When $\boldsymbol{x} = \boldsymbol{3}$ and $\boldsymbol{a} = \boldsymbol{5}$: $S_n = e^{3ln5} = 125 = \boldsymbol{5^3}$
When $\boldsymbol{x} = \boldsymbol{4}$ and $\boldsymbol{a} = \boldsymbol{5}$: $S_n = e^{4ln5} = 625 = \boldsymbol{5^4}$

➔ That is so interesting, that I noticed :

- $T_n\ (a,x)$ increases when *a* or/and *x* increases.
- The sum of terms of this infinite sequence equals $e^{x\ln a}$, which then I realised is equal $\mathbf{a^x}$.

To test more about this with other values of *a* and *x*, I need to take some other values of *a* and *x*, then calculate $T_n\ (a,x)$ to check the validity of general statement written above.

So let's take **$a = 1.5$ and $x = 8$**

$a = 1.5$ and $x = 9$

$a = 1.5$ and $x = 10$

$a = 1.5$ and $x = 11$

$a = 1.5$ and $x = 12$

^ Values of *a* and *x* are positive, *x* is increasing.

I will calculate 3 different sums (T_3, T_4 and T_{10}) in these 5 cases (*8, 9, 10, 11, 12*).

When **$a = 1.5$ and $x = 8$**:

$T_3\ (1.5, 8) = t_0 + t_1 + t_2$

$$= 1 + \frac{8\ln 1.5}{1!} + \frac{(8\ln 1.5)^2}{2!}$$

$$= 9.504583$$

$T_4\ (1.5, 8) = t_0 + t_1 + t_2 + t_3$

$$= 1 + \frac{8\ln 1.5}{1!} + \frac{(8\ln 1.5)^2}{2!} + \frac{(8\ln 1.5)^3}{3!}$$

$$= 15.192839$$

$T_{10}\ (1.5, 8) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9$

$$= 1 + \frac{8\ln 1.5}{1!} + \frac{(8\ln 1.5)^2}{2!} + \frac{(8\ln 1.5)^3}{3!} + \frac{(8\ln 1.5)^4}{4!} + \frac{(8\ln 1.5)^5}{5!} + \frac{(8\ln 1.5)^6}{6!} + \frac{(8\ln 1.5)^7}{7!} + \frac{(8\ln 1.5)^8}{8!} + \frac{(8\ln 1.5)^9}{9!}$$

$$= 25.579143$$

When **$a = 1.5$ and $x = 9$**:

$T_3\ (1.5, 9) = t_0 + t_1 + t_2$

$= 1 + \frac{9\ln 1.5}{1!} + \frac{(9\ln 1.5)^2}{2!}$

$= 11.307465$

$T_4\,(1.5, 9) = t_0 + t_1 + t_2 + t_3$

$= 1 + \frac{9\ln 1.5}{1!} + \frac{(9\ln 1.5)^2}{2!} + \frac{(9\ln 1.5)^3}{3!}$

$= 19.406564$

$T_{10}\,(1.5, 9) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9$

$=$

$1 + \frac{9\ln 1.5}{1!} + \frac{(9\ln 1.5)^2}{2!} + \frac{(9\ln 1.5)^3}{3!} + \frac{(9\ln 1.5)^4}{4!} + \frac{(9\ln 1.5)^5}{5!} + \frac{(9\ln 1.5)^6}{6!} + \frac{(9\ln 1.5)^7}{7!}$
$+ \frac{(9\ln 1.5)^8}{8!} + \frac{(9\ln 1.5)^9}{9!}$

$= 38.273650$

When ***a* = *1.5* and *x* = *10***:

$T_3\,(1.5, 10) = t_0 + t_1 + t_2$

$= 1 + \frac{10\ln 1.5}{1!} + \frac{(10\ln 1.5)^2}{2!}$

$= 13.274749$

$T_4\,(1.5, 10) = t_0 + t_1 + t_2 + t_3$

$= 1 + \frac{10\ln 1.5}{1!} + \frac{(10\ln 1.5)^2}{2!} + \frac{(10\ln 1.5)^3}{3!}$

$= 24.384625$

$T_{10}\,(1.5, 10) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9$

$=$

$1 + \frac{10\ln 1.5}{1!} + \frac{(10\ln 1.5)^2}{2!} + \frac{(10\ln 1.5)^3}{3!} + \frac{(10\ln 1.5)^4}{4!} + \frac{(10\ln 1.5)^5}{5!} + \frac{(10\ln 1.5)^6}{6!} + \frac{(10\ln 1.5)^7}{7!}$
$+ \frac{(10\ln 1.5)^8}{8!} + \frac{(10\ln 1.5)^9}{9!}$

$= 57.152951$

When ***a* = *1.5* and *x* = *11***:

$T_3\,(1.5, 11) = t_0 + t_1 + t_2$

$$= 1 + \frac{11\ln 1.5}{1!} + \frac{(11\ln 1.5)^2}{2!}$$

$$= 15.406434$$

$$T_4\,(1.5, 11) = t_0 + t_1 + t_2 + t_3$$

$$= 1 + \frac{11\ln 1.5}{1!} + \frac{(11\ln 1.5)^2}{2!} + \frac{(11\ln 1.5)^3}{3!}$$

$$= 30.193679$$

$$T_{10}\,(1.5, 11) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9$$

$$= 1 + \frac{11\ln 1.5}{1!} + \frac{(11\ln 1.5)^2}{2!} + \frac{(11\ln 1.5)^3}{3!} + \frac{(11\ln 1.5)^4}{4!} + \frac{(11\ln 1.5)^5}{5!} + \frac{(11\ln 1.5)^6}{6!}$$

$$+ \frac{(11\ln 1.5)^7}{7!} + \frac{(11\ln 1.5)^8}{8!} + \frac{(11\ln 1.5)^9}{9!}$$

$$= 85.097411$$

When ***a* = *1.5* and *x* = *12***:

$$T_3\,(1.5, 12) = t_0 + t_1 + t_2$$

$$= 1 + \frac{12\ln 1.5}{1!} + \frac{(12\ln 1.5)^2}{2!}$$

$$= 17.702522$$

$$T_4\,(1.5, 12) = t_0 + t_1 + t_2 + t_3$$

$$= 1 + \frac{12\ln 1.5}{1!} + \frac{(12\ln 1.5)^2}{2!} + \frac{(12\ln 1.5)^3}{3!}$$

$$= 36.900388$$

$$T_{10}\,(1.5, 12) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9$$

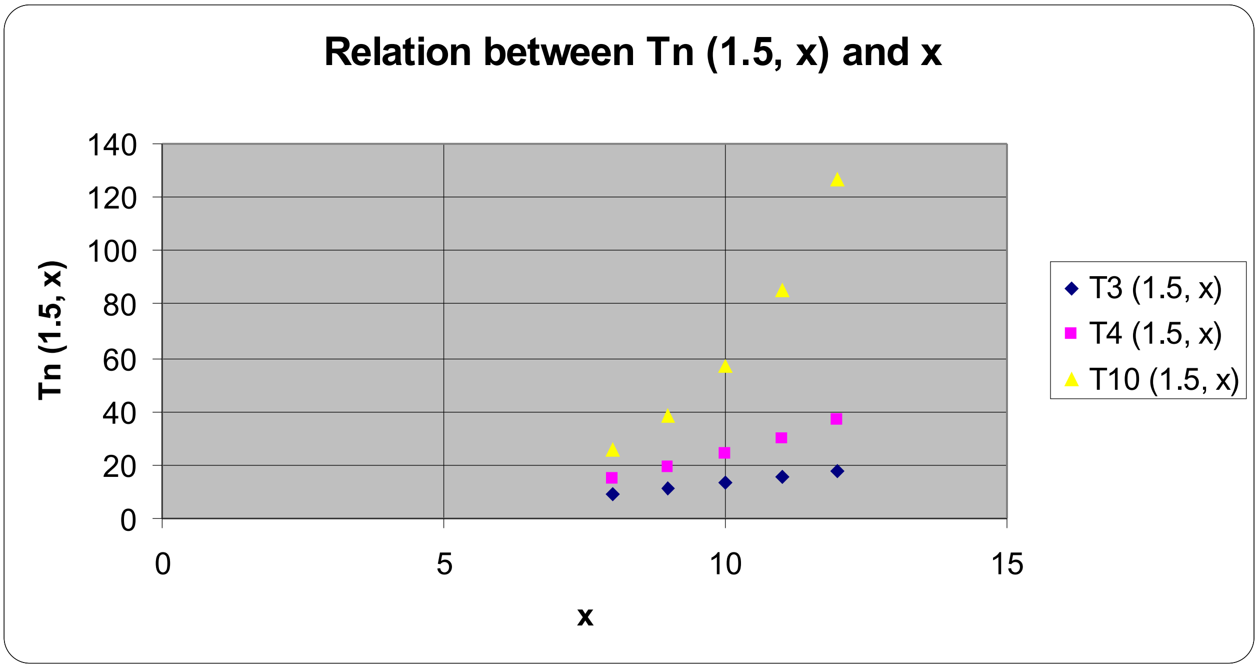
$$= 1 + \frac{12\ln 1.5}{1!} + \frac{(12\ln 1.5)^2}{2!} + \frac{(12\ln 1.5)^3}{3!} + \frac{(12\ln 1.5)^4}{4!} + \frac{(12\ln 1.5)^5}{5!} + \frac{(12\ln 1.5)^6}{6!}$$

$$+ \frac{(12\ln 1.5)^7}{7!} + \frac{(12\ln 1.5)^8}{8!} + \frac{(12\ln 1.5)^9}{9!}$$

$$= 126.215786$$

Now, using Microsoft Excel, I plot the relation between T_3, T_4, T_{10} and x:

	x = 8	x = 9	x = 10	x = 11	x = 12
T_3 (1.5, x)	9.504583	11.307465	13.274749	15.406434	17.702522
T_4 (1.5, x)	15.192839	19.406564	24.384625	30.193679	36.900388
T_{10} (1.5, x)	25.579143	38.273650	57.152951	85.097411	126.215786



So after testing what I have noticed before, here I still see that when *a* and *x* are positive, if value of *x* increases, $T_n(a,x)$ also increases.

So what if *x* is negative? To be sure that the statement is true in all cases, there is a need to also test some negative values of *x*. Let's take $a = 2$, and 4 different negative values of *x*: *-8, -7, -3, -2* (ascending order). Using GDC, I will calculate $T_9\ (2,x)$ and see the relation between $T_9\ (2,x)$ and *x*.

When ***a* = 2 and *x* = -8**:

$$T_9\ (2, -8) = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8$$

$$= 1 + \frac{-8\ln 2}{1!} + \frac{(-8\ln 2)^2}{2!} + \frac{(-8\ln 2)^3}{3!} + \frac{(-8\ln 2)^4}{4!} + \frac{(-8\ln 2)^5}{5!} + \frac{(-8\ln 2)^6}{6!} + \frac{(-8\ln 2)^7}{7!} + \frac{(-8\ln 2)^8}{8!}$$

$$= 8.679707$$

When **a = 2 and x = -7**:

$$T_9(2, -7) = 1 + \frac{-7\ln 2}{1!} + \frac{(-7\ln 2)^2}{2!} + \frac{(-7\ln 2)^3}{3!} + \frac{(-7\ln 2)^4}{4!} + \frac{(-7\ln 2)^5}{5!} + \frac{(-7\ln 2)^6}{6!} + \frac{(-7\ln 2)^7}{7!} + \frac{(-7\ln 2)^8}{8!}$$

$$= 2.743859$$

When **a = 2 and x = -3**:

$$T_9(2, -3) = 1 + \frac{-3\ln 2}{1!} + \frac{(-3\ln 2)^2}{2!} + \frac{(-3\ln 2)^3}{3!} + \frac{(-3\ln 2)^4}{4!} + \frac{(-3\ln 2)^5}{5!} + \frac{(-3\ln 2)^6}{6!} + \frac{(-3\ln 2)^7}{7!} + \frac{(-3\ln 2)^8}{8!}$$

$$= 0.016654$$

When **a = 2 and x = -2**:

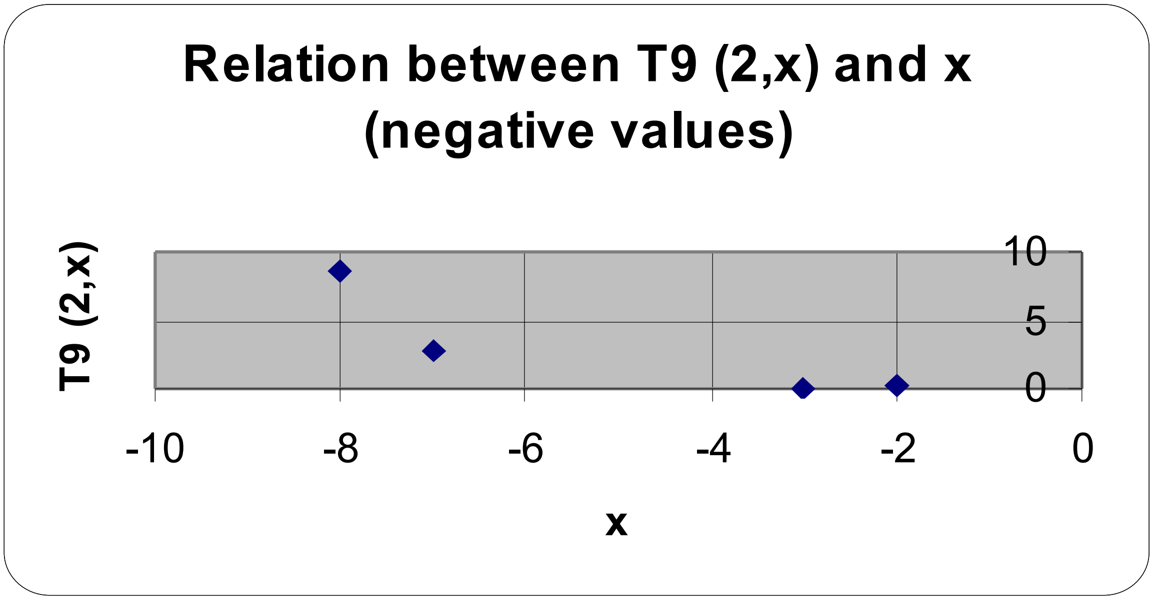
$$T_9(2, -2) = 1 + \frac{-2\ln 2}{1!} + \frac{(-2\ln 2)^2}{2!} + \frac{(-2\ln 2)^3}{3!} + \frac{(-2\ln 2)^4}{4!} + \frac{(-2\ln 2)^5}{5!} + \frac{(-2\ln 2)^6}{6!} + \frac{(-2\ln 2)^7}{7!} + \frac{(-2\ln 2)^8}{8!}$$

$$= 0.250046$$

Using Microsoft Excel, I will plot the relation between $T_9(2,x)$ and x (negative) :

x	y
-8	8.679707
-7	2.743859

-3	0.016654
-2	0.250046



So, here I haven't noticed any sign of similarity to the cases before with *x*-positive. As *x* increases, T_n decreases then increases. So for the general statement found above, we have to note that it's untrue for *x*-negative.

Now, let's point out a little bit about the scope of the general statement:

$$\sum_{n=0}^{\infty} \frac{(x\ln a)^n}{n!} = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \frac{(x\ln a)^4}{4!} + \ldots = e^{x\ln a} = a^x$$

- We know the range of values $T_n(a,x)$.
- We know how $T_n(a,x)$ changes when x changes.
- We know the domain for a and x: positive numbers.
- It's easy to find the infinite sum, just by setting values for a and x.
- We know this is power series expansion.

What I did to find out this statement is just calculating, and while doing this, I have been realising, step by step, some signs that suggest about the $T_n (a,x)$, like range, sign and simplest formula.

After doing this portfolio, I learned several things, such as

- using mathematical technology on computer, which I did not know before;
- constructing the parts of the work so that it looks logically;
- using appropriate language when doing mathematical big work;
- realising subtleties from graph/plot rather than from statistics as before I was used to;
- how to find the sum of infinite general sequence.

I DECLARE THAT THE WHOLE WORK IS ENTIRELY DONE ON MY OWN.

THE END