PORTFOLIO TYPE I – MATHEMATICAL INVESTIGATION
-MATRIX BINOMIALS-

Student: Domazet Neven SL

In this portfolio,my assignment is to deal with matrix binomials,and to investigate them. At first,my initial matrices are as follows:

$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Now,my task is to find and calculate following matrices: : X^2, X^3, X^4; Y^2,Y3 and Y^4

So,since $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $X^2 = X * X$ we will calculate that as $X^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

We know that we can express multiplication of matrices as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} * \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$,and it goes like

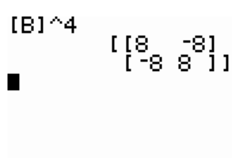
$\begin{pmatrix} a_1 \cdot b_1 + a_2 \cdot b_3 & a_1 \cdot b_2 + a_2 \cdot b_4 \\ a_3 \cdot b_1 + a_4 \cdot b_3 & a_3 \cdot b_2 + a_4 \cdot b_4 \end{pmatrix}$,so we get that

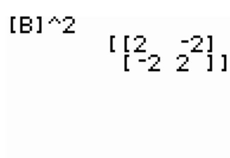
$X^2 = \begin{pmatrix} 1*1+1*1 & 1*1+1*1 \\ 1*1+1*1 & 1*1+1*1 \end{pmatrix}$, thus $X^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Or 2* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$=2* X

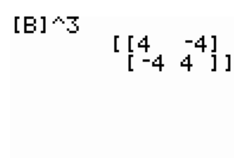
As we now know this,it is easy to calculate X^3 ,because $X^3 = X^2 * X$ or 2*X*X,which is 2*2*X or it is equal to 2^2*X and equal to $X^3 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$.

X^4 is defined as X^3*X=2^2 *X*X=2^2 *2*X=2^3 *X and our solution is $X^4 = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$

Procedure for Y is completely the same as for X,and we do not have to go in details,so to calculate the products of Y^2, Y^3 and Y^4, we will use the TI-84 Plus Texas Instruments Graphic Calculator and obtain the results in the following form ([B] equals Y)







The results are as follows: $Y^2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$; $Y^3 = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$ and $Y^4 = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$.

Now to continue with our work we need our previous results,because they can help us to find X^n, Y^n and $(X+Y)^n$ .Results are shown in table below

$$X=\begin{pmatrix}1&1\\1&1\end{pmatrix} \qquad Y=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$
$$X^2=\begin{pmatrix}2&2\\2&2\end{pmatrix} \qquad Y^2=\begin{pmatrix}2&-2\\-2&2\end{pmatrix}$$
$$X^3=\begin{pmatrix}4&4\\4&4\end{pmatrix} \qquad Y^3=\begin{pmatrix}4&-4\\-4&4\end{pmatrix}$$
$$X^4=\begin{pmatrix}8&8\\8&8\end{pmatrix} \qquad Y^4=\begin{pmatrix}8&-8\\-8&8\end{pmatrix}$$

Here we can see some characteristics which can help us in our work,and we will stick to them,and applying the rule to the results we will get the following matrix identities:

$$X=\begin{pmatrix}1&1\\1&1\end{pmatrix};\quad X^2=2*\begin{pmatrix}1&1\\1&1\end{pmatrix};\quad X^3=4*\begin{pmatrix}1&1\\1&1\end{pmatrix};\quad X^4=8*\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

If we express every coefficient as 2^a ,we get $X=2^0*\begin{pmatrix}1&1\\1&1\end{pmatrix}$; $X^2=2^1*\begin{pmatrix}1&1\\1&1\end{pmatrix}$;

$X^4=2^3*\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$; $X^4=2^3*\begin{pmatrix}1&1\\1&1\end{pmatrix}$.

Next table that can help us is the one dealing with relation between the powers

$$\begin{array}{cc} n=1 & a=0\\ n=2 & a=1\\ n=3 & a=2\\ n=4 & a=3\end{array}$$

It is visible now that that $a=n-1$

To prove it I will use an example in which $n=12$. Hence

$$2^{12-1} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$2048 * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2048 & -2048 \\ -2048 & 2048 \end{pmatrix}$$

To show that my calculations were correct I will once again use the TI-84 GDC (NOTE: [A] = Y)

Now I will find the formula for the general term $(X+Y)^n$ assuming that $(X+Y)^n = X^n + Y^n$ and taking like an example that $n = 3$:

$X^3 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ and $Y^3 = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

$X^3 - Y^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$

Also $(X+Y)^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$

This proves that $(X+Y)^n = X^n + Y^n$ and since we know that $X^n = 2^{n-1} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y^n = 2^{n-1} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ this means that $(X+Y)^n = 2^{n-1} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2^{n-1} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Now we can calculate $(X+Y)^n$

$$(X+Y)^n = 2^{n-1} * (X+Y)$$

$$(X+Y)^n = 2^{n-1} * \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(X+Y)^n = 2^{n-1} * 2^1 * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(X+Y)^n = 2^{n-1+1} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
[A]^12
 [[2048   -2048]
  [-2048 2048 ]]
```

$$(X+Y)^n = 2^n * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we can now see an expression for matrix as X^n and substituting $a = n-1$ we can find a general form $X^n = 2^{n-1}\mathbf{X}$ also same goes for Y as $Y^n = 2^{n-1}\mathbf{Y}$

Now when we have found X^n and Y^n as expressions our next task is to find $(X+Y)^n$,and it is quite simple:
Let us first sum up X+Y matrices and it goes like

X+Y=$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ or 2* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and it is now obvious that this is in form of 2*E,as E stands for

Now when we now this it goes like

$(X+Y)^n$ =(2*E)^n =2^n *E^n =2^n *$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$=$\begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$ because E^n =E

Following these general expressions, we may continue to investigate the pattern in the following equalities $A = aX$ and $B = bY$, when a and b are constants and they have different values.

It is very important to mention that X*Y=$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and it is obvious that X*Y=0 and 0 stands for zero matrix

Now we can calculate A^2 =(aX)^2 =a^2 *X*X=2 *a^2 *X

A^3 = A^2 *A=2 *a^2 *X*(aX)= 2^3 * X$^{2=}$ 2^2 * a * a^n *X

A^4 = A^3 *A= 2^3 * a^4 *X

For B it is the same so

B^2 =(bX)^2 =b^2 *X*X=2 *b^2 *X

B^3 = B^2 *B=2 *b^2 *X*(bX)= 2^3 * X$^{2=}$ 2^2 * b^3 *X

B^4 = B^3 *B= 2^3 * b^4 *X

Or we can put some numbers,for example a=4 and b=5 and now we have

$$A = 4 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

$$A^2 = 4^2 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = 16 * \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix}$$

$$A^3 = 4^3 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^3 = 64 * \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 256 & 256 \\ 256 & 256 \end{pmatrix}$$

$$A^4 = 4^4 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^4 = 256 * \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = \begin{pmatrix} 2048 & 2048 \\ 2048 & 2048 \end{pmatrix}$$

$$B = 5 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$B^2 = 5^2 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = 25 * \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix}$$

$$B^3 = 5^3 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^3 = 125 * \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 500 & 500 \\ 500 & 500 \end{pmatrix}$$

$$B^4 = 5^4 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^4 = 625 * \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = \begin{pmatrix} 5000 & 5000 \\ 5000 & 5000 \end{pmatrix}$$

Now we can see some characteristics and substitute powers instead of numbers and get

$A^n = 2^{n-1} * a^n *$X

We saw in earlier example that it is the same with B and it simply goes like

$B^n = 2^{n-1} * b^n *$X

We proved that X*Y=0 and commutatively it means that A*B=0 so general expression goes like

$(A+B)^n = A^n + B^n = 2^{n-1}(a^n * X + b^n * Y)$

Now we have to consider matrix M= $\begin{pmatrix} a\ +b\ + & a-b \\ a\ -b & a\ +b \end{pmatrix}$

We now from previous examples that $A = aX$ and $B = bY$.

Or A=a * $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and B=b* $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ so adding them we get

A+B= $\begin{pmatrix} a & a \\ a & a \end{pmatrix} + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix} = \begin{pmatrix} a\ +b\ + & a-b \\ a\ -b & a\ +b \end{pmatrix}$ =M so we proved now that

A+B=M

Or with numbers form previous example,we get that

A=$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ and B=$\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ and From the formula $M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$

we can calculate matrix M by substituting values for a and b:

$$M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

$$M = \begin{pmatrix} 4+5 & 4-5 \\ 4-5 & 4+5 \end{pmatrix}$$

$$M = \begin{pmatrix} 9 & -1 \\ -1 & 9 \end{pmatrix}$$

And now I will calculate the sum of matrices A and B:

$$A+B = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} = \begin{pmatrix} 9 & -1 \\ -1 & 9 \end{pmatrix}$$

This proves the formula M = A + B while $M = \begin{pmatrix} 9 & -1 \\ -1 & 9 \end{pmatrix}$ and $M = \begin{pmatrix} 9 & -1 \\ -1 & 9 \end{pmatrix}$. I will take another example but this time using negative numbers as constants a and b to prove whether this formula is correct for those numbers as well. So in this example $a = -2$ and $b = -3$. Then $A = -2 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$ $B = -3 * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$

Then we calculate $A+B = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} + \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}$ and also

$M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$; $M = \begin{pmatrix} -2+(-3) & -2-(-3) \\ -2-(-3) & -2+(-3) \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}$. Like in the previous example M = A + B which proves this formula also for the negative numbers.

Similarly we can prove first that M^2=$\begin{pmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{pmatrix} + \begin{pmatrix} a + b + & a-b \\ a - b & a + b \end{pmatrix}$

and we see that this is $\begin{pmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{pmatrix}$ and when we simplified this like

2* $\begin{pmatrix} a^2 & a^2 \\ a^2 & a^2 \end{pmatrix}$+2 $\begin{pmatrix} b^2 & b^2 \\ b^2 & b^2 \end{pmatrix}$ and we now see that M$^{2=}$ $A^2 + B^2$,it is completely the same for numbers,as we have shown for M=A+B

There is also other way to prove this,as we now that $M^2 = (A+B)*(A-B)$,and we earlier proved that A*B=0(zero matrix),now it goes easily like

$M^2 = (A+B)*(A-B) = A^2 + A*B + B*A + B^2 = A^2 + 0 + 0 + B^2 = A^2 + B^2$

We also already proved that $2^{2-1} * 2^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-3)^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = A^n + B^n$

And now if $M^n = (A+B)^n$ we see that

$M^n = 2^{n-1}(a^n * X + b^n * Y)$

By taking for example **a=2** **b=-3** **n=2**

$$(A+B)^2 = \begin{pmatrix} 2 - 3 & 2-(-3) \\ 2 - (-3) & 2-3 \end{pmatrix}^2 = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix} * \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 26 & -10 \\ -10 & 26 \end{pmatrix}$$

$$2^{2-1} * 2^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-3)^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2*(\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}) = 2*\begin{pmatrix} 13 & -5 \\ -5 & 13 \end{pmatrix} = \begin{pmatrix} 26 & -10 \\ -10 & 26 \end{pmatrix}$$

Through my work,I have shown that a and b belong to the set of rational number Q and n belongs to set of natural numbers $\square_0$.

I have also shown that $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ when multiplied give a zero matrix:

$X * Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.and that A*B gives zero matrix as an in example with

$(A+B)^2 = A^2 + AB + BA + B^2$ in which like

$(aX+bY)^2 = a^2X^2 + b^2Y^2 + [abXY + abYX]$ and XY gives zero so that

$(aX+bY)^2 = a^2X^2 + b^2Y^2$

Because we know that M=A+B than referring to the law of exponentiation we see that

$M^n = 2^{n-1}(a^n * X + b^n * Y)$