Mathematics SL Portfolio

Type I: Mathematical Investigation

Logarithm Bases

Mathematical Investigation: Logarithm Bases

In this investigation I will be examining logarithms and their bases. The purpose of examining logarithmic bases is to find patterns between the bases, its exponents and the product. Throughout this investigation I will be looking at different sets of logarithms and trying to determine a pattern. After I have found a pattern I will test its validity by applying the pattern to other sets of logarithms. First I'm going to look at four sequences of logarithms and then continue the pattern for two more terms.

1. $\mathbf{\log_2 8,\ \log_4 8,\ \log_8 8,\ \log_{16} 8,\ \log_{32} 8,}$ $\underline{\log_{64} 8,\ \log_{128} 8}$
2. $\mathbf{\log_3 81,\ \log_9 81,\ \log_{27} 81,\ \log_{81} 81,}$ $\underline{\log_{243} 81,\ \log_{729} 81}$
3. $\mathbf{\log_5 25,\ \log_{25} 25,\ \log_{125} 25,\ \log_{625} 25,}$ $\underline{\log_{3125} 25,\ \log_{15625} 25}$
4. $\mathbf{\log_m m^k,\ \log_{m^2} m^k,\ \log_{m^3} m^k,\ \log_{m^4} m^k,}$ $\underline{\log_{m^5} m^k,\ \log_{m^6} m^k}$

After studying the four sequences I noticed a similar pattern in each of them. In sequence 1 each term increased by multiplying 2 to the previous term's base. But to express the sequence in terms on the term number, I realized that it could simply be viewed as the base of 2 to the power of the term number. The other sequences also follow a similar pattern. Sequence 2 can be written as a base of 3 to the power of n and sequence four also can be written as a base of m to the power of n. An expression to calculate the n^{th} term of each of the four sequences are written below in the form $\frac{p}{q}$, where p, q $\in$ Z.

1. $\dfrac{\log 8}{\log 2}$

2. $\dfrac{\log 81}{\log 3}$

3. $\dfrac{\log 25}{\log 5}$

4. $\frac{\log m}{\log m}$

The expressions above can also be proven using a graph. For each sequence, I can graph each term as well as the expression determined for the nth term of that sequence on the same axis. The point of intersection between the graph of each term and the graph of the nth term would have a special connection. The x coordinate of the point of intersection would reflect the term number and the y coordinate of the point of intersection would reflect the value when the term is evaluated. Thus, the expressions above can also be proven using a graph.

Figure 1.1 shows 4 graphs and their points of intersection.

Y= s(x), where s(x) = $\frac{\log 8}{\log 2}$

Y= t(x), where t(x) = $\log_2 8$

Y = u(x), where u(x) = $\log_{16} 8$

Y = v(x), where v(x) = $\log_8 8$

These graphs support the expression $\frac{\log 8}{\log 2}$ and their points of intersection also fit the theory mentioned above.

Figure 1.2 shows 4 graphs and their points of intersection

Y = f(x), where f(x) = $\frac{\log 81}{\log 3}$

Y = h(x), where h(x) = $\log_3 81$

Y = q(x), where q(x) = $\log_9 81$

Y = r(x), where r(x) = $\log_{27} 81$

These graphs also support the expression $\frac{\log 81}{\log 3}$ and the points of intersection also fit the theory mentioned above.

Figure 1.3 shows 4 graphs and its points of intersection

Y = f(x), where f(x) = $\log_5 25$

Y = g(x), where g(x) = $\frac{\log 25}{\log 5}$

Y = h(x), where h(x) = $\log_{125} 25$

Y= s(x), where s(x) = $\log_{25} 25$

These graphs support the expression $\frac{\log 25}{\log 5}$ and its point of intersection also fit the theory mentioned above.

I will now calculate the values for other sequences and try to find a pattern on how to obtain the answers. All answers are written in the form $\frac{p}{q}$, where p, q $\in$ Z.

1. $\log_4 64$ $\log_8 64$ $\log_{32} 64$

   $= 3$ $= 2$ $= \frac{6}{5}$

2. $\log_7 49$ $\log_{49} 49$ $\log_{343} 49$

   $= 2$ $= 1$ $= \frac{2}{3}$

3. $\log_{\frac{1}{5}} 125$ $\log_{\frac{1}{125}} 125$ $\log_{\frac{1}{625}} 125$

   $= -3$ $= -1$ $= -\frac{3}{4}$

4. $\log_8 512$ $\log_2 512$ $\log_{16} 512$

   $= 3$ $= 9$ $= \frac{9}{4}$

After analyzing the above sequences, I noticed a pattern between the first two answers and the third answer. To find the third answer it is necessary to multiply the first two answers. The value after multiplying the first two answers produces the numerator for the third answer. To find the denominator for the third answer it is necessary to add up the first two answers. Thus, the third answer is usually written in the form $\frac{p}{q}$ which in some cases can be simplified further. To prove this theory I will create two more sequences which follow the same pattern as the sequences above. Then I will evaluate the terms to prove my theory.

1. $\log_6 216$ $= 3$ $\log_{36} 216$ $= 1.5$ or $\frac{3}{2}$ $\log_{216} 216$ $= 1$

$$\frac{3(1.5)}{3+1.5} = 1$$

2. $\log_9 729$ $= 3$ $\log_3 729$ $= 6$ $\log_{27} 729$ $= 2$

$$\frac{3(6)}{3+6} = \frac{18}{9} = 2$$

The pattern shown above can be generalized for most logarithms. So, I will create a general statement for logarithms following this pattern.

Let $\mathbf{\log_a x = c}$ and $\mathbf{\log_b x = d}$
So, $\mathbf{\log_{ab} x} = \frac{cd}{c+d}$

Basic exponent and logarithm properties can be used to derive the general statement.

Let $\mathbf{\log_a x = c}$ Let $\mathbf{\log_b x = d}$

By using a logarithm property we can convert these logs into exponents:

$a^c = x$ $b^d = x$

$(a^c)^d = x^d$ $(b^d)^c = x^c$

Then we can multiply both equations:

$a^{cd} = x^d$ $b^{dc} = x^c$

$(ab)^{cd} = x^{c+d}$

Now we can use log properties to simplify the equation:

cd log(ab) = c+d log(x)

$$\frac{cd}{c+d} = \frac{\log x}{\log ab}$$

$$\frac{cd}{c+d} = \log_{ab}x$$

So, the general statement is found $\frac{cd}{c+d} = \log_{ab}x$

To test the validity of my general statement I will use other values of a, b and x to see if my results match the general statement.

1. $\log_{16}4096 = 3$ $\quad \log_{4}4096 = 6$ $\quad \log_{64}4096 = \frac{3(6)}{3+6} = \frac{18}{9} = 2$

2. $\log_{12}1728 = 3$ $\quad \log_{6}1728 = 4.16$ $\quad \log_{72}1728 = \frac{3(4.16)}{3+4.16} = \frac{12.48}{7.16} \approx 1.74$

3. $\log_{27}729 = 2$ $\quad \log_{3}729 = 6$ $\quad \log_{81}729 = \frac{2(6)}{2+6} = \frac{12}{8} = \frac{3}{2}$

4. $\log_{\frac{1}{49}} 2401$ $\qquad$ $\log_{\frac{1}{7}} 2401$ $\qquad$ $\log_{\frac{1}{343}} 2401$

$$= \frac{-2(-4)}{-2+(-4)}$$

$= -2$ $\qquad$ $= -4$ $\qquad$ $= -\frac{8}{6}$

$$= -\frac{4}{3}$$

5. $\log_{\frac{1}{144}} 20736$ $\qquad$ $\log_{\frac{1}{12}} 20736$ $\qquad$ $\log_{\frac{1}{1728}} 20736$

$$= \frac{-2(-4)}{-2+(-4)}$$

$= -2$ $\qquad$ $= -4$ $\qquad$ $= -\frac{8}{6}$

$$= -\frac{4}{3}$$

After testing the validity of the general statement, I realized that the statement does not work for all numbers. There are some limitations for the values of a, b and x. the first limitation is that all values of a, b and x must be greater than 0. The values must be greater than 0 because it is not possible to calculate the log of a negative number or 0. In other words the values of a, b and x must be all real numbers. Another limitation to this statement is that the sum of a+b must be greater than 0. This restriction applies because if the values of a+b are less than 0 then you have an error in the domain. The final restriction is that the value of a can not be $\frac{1}{b}$. This applies to the general statement because it is impossible to divide a number by 0. Thus, these are all the limitations on the general statement.

Throughout this portfolio I have investigated the patterns between logarithms and their bases. To find these patterns I looked at different sequences of logarithms. Then I found relationships between the values of the first two terms and how that value relates to the third term. By following this process I was able to find a general statement and test its validity. Then I described the limitations as well as the scope of the general statement. So, by completing this portfolio I was able to understand a lot about logarithms, their bases and the relationships between them.

Fig. 1.1

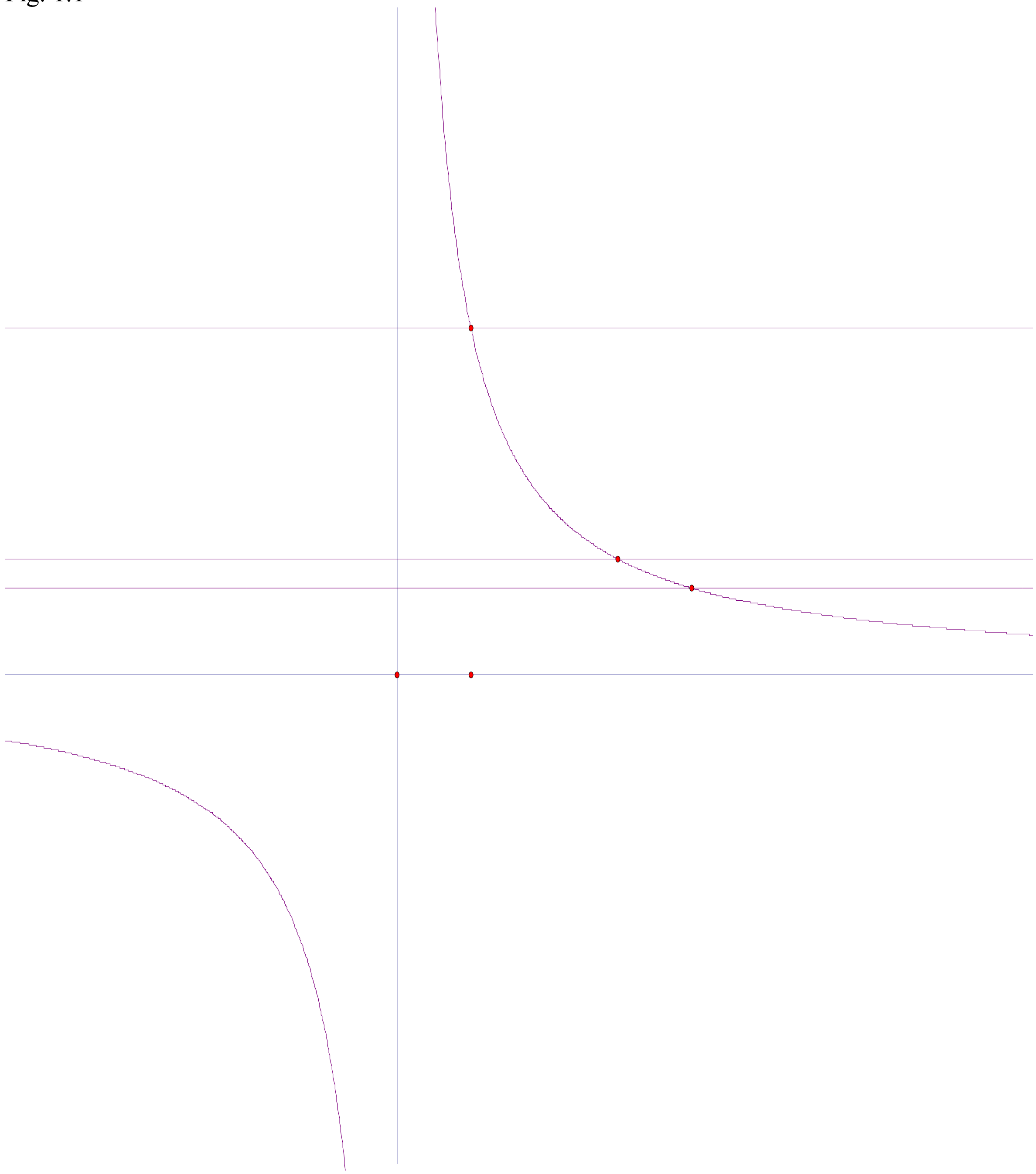

Fig. 1.2

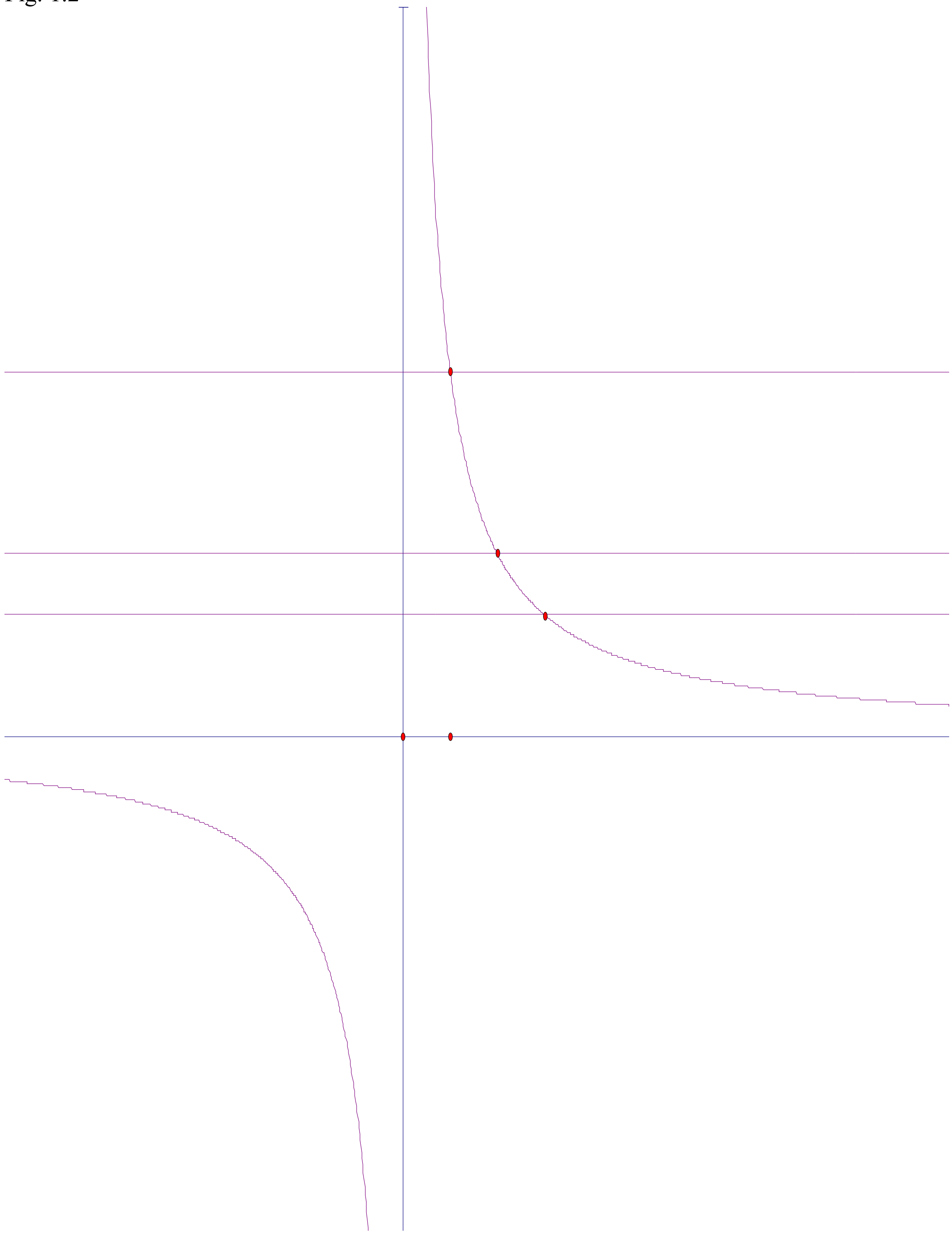

Fig. 1.3