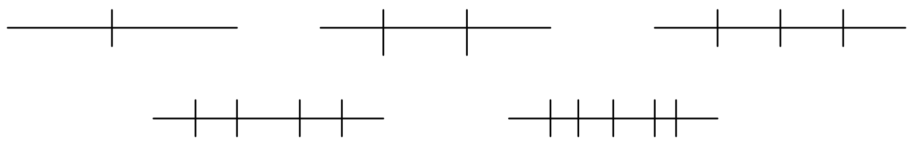
How many pieces?

Mathematics can be considered to be the study of patterns. A useful ability in math can be forming a rule to describe a pattern. Of course any rule that we develop must be true in all relevant cases. In this essay, I am going to investigate the maximum number of pieces obtained when n-dimensional object is cut, and then prove it is true.

◆ One-dimensional object

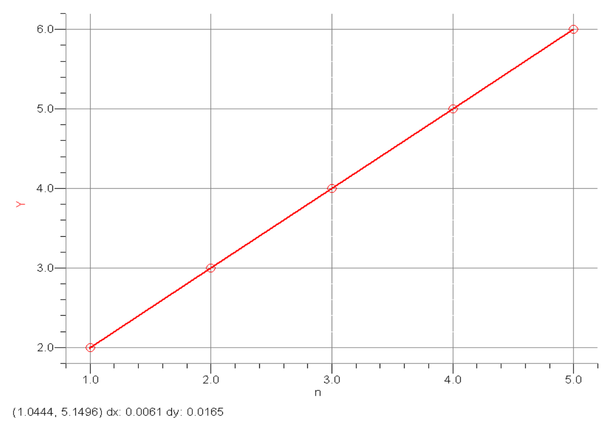
Processing: Find the maximum number of segments that can be cut in n-cuts of a line segment. To begin the solution, consider the results for n=1, 2,3,4,5.



Let **S** represent the maximum number of segments in n-cuts of a line segment. The value for **S** is shown in the table.

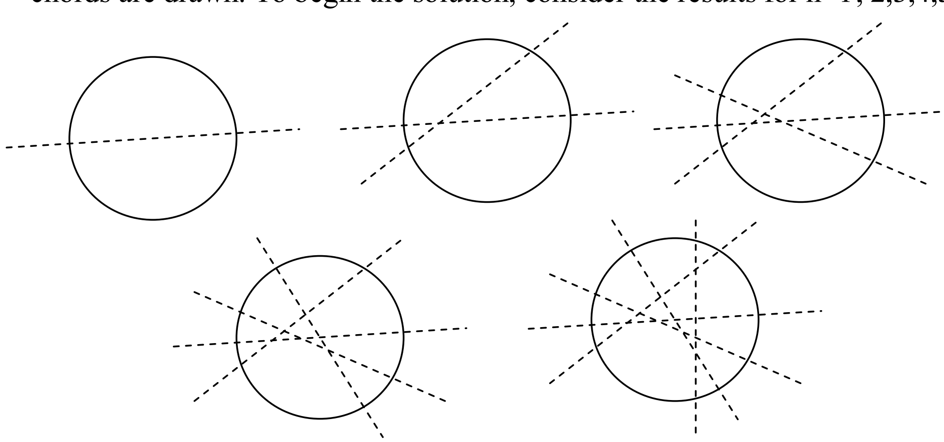
n	1	2	3	4	5
S	2	3	4	5	6

Plotting the points related to the variables n and S, suggests that the relationship between them could be linear, and so we might assume that S=kn+1.
Substituting the first value for n gives:
n=1, 2=k+1, k=1
Therefore, the rule which related the maximum number of segments obtained from n cuts is S=n+1.



◆ Two-dimensional object

Processing: Find the maximum number of regions that can be obtained when n chords are drawn. To begin the solution, consider the results for n=1, 2,3,4,5.



Let **R** represent the maximum number of pieces in n-cuts of a line segment. The value for **R** is shown in the table.

n	1	2	3	4	5
R	2	4	7	11	16

①Recursive rule:

$R_1=2$
$R_2=4=2+2$
$R_3=7=4+3$
$R_4=11=7+4$
$R_5=16=11+5$
…
$R_n=R_{n-1}+n$

Assume that $R_0=1$, then $R_n=1+1+2+3+\cdots\cdots+n$

$R_n=1+(1+2+3+\cdots\cdots+n)$

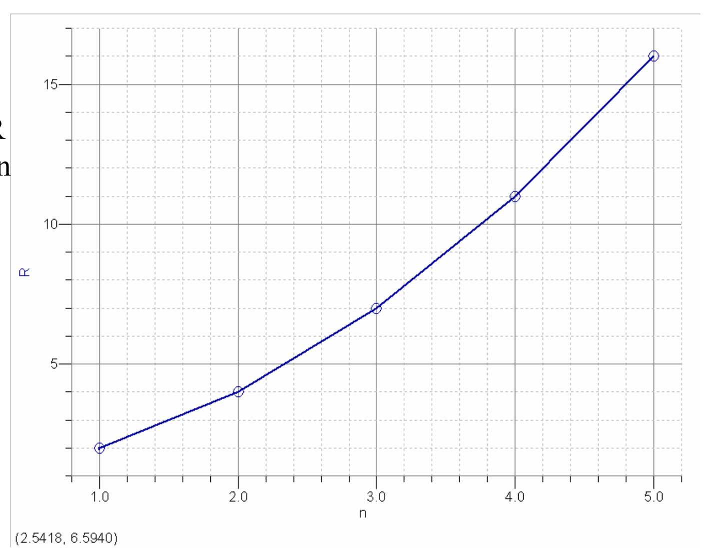
$=1+2+3+\cdots\cdots+n=n/2[2a+(n-1)*d]$, $a=1$, $d=1$

$\Rightarrow 1+2+3+\cdots\cdots+n=n/2[2+(n-1)]=n/2[n+1]=(n^2+n)/2$

Therefore, the rule to generate the maximum number of regions is
$1+ (n^2+n)/2=$ **$(n^2+n+2)/2$.**
When n=5, $R_5=(5^2+5+2)/2=16$, which corresponds to the tabulated value for n=5 above.

$R=(n^2+n+2)/2$

②A conjecture for the relationship between the maximum number of regions(R) and the number of chords(n).



Use graphical analysis to sketch the graph related to the variables n and R suggests that the relationship between them could be quadratic, and so we might assume that

$R=an^2+bn+c$

Substituting the first three values for n gives:
$n=1 \Rightarrow 2=a+b+c$
$n=2 \Rightarrow 4=4a+2b+c$
$n=3 \Rightarrow 7=9a+3b+c$

Solve these three equations for
a, b, c gives
a=1/2, b=1/2, c=1

Thus R_n =1/2n^2 +1/2n+1

When n=5, R_5=1/2*5^2+1/2*5+1=16, which is also corresponds to the tabulated value for n=5 above.

R_n =1/2n^2 +1/2n+1

R_n =1/2n^2 +1/2n+1 is the same with R_n=(n^2+n+2)/2
So far we have formed a conjecture that the maximum number of regions from n-chords is given by R= (n^2+n+2)/2.

Proof:
Let P(n) be the proposition that the maximum number of regions that can be separated from n-chords is given by R= (n^2+n+2)/2 for $n \geq 0$

Step1: P(n) is true for n=1 as R= (1^2+1+2)/2=2 which is the maximum number of regions from 1 chord.
Step2: Assume that P(n) is true for a k-chords circle i.e., that R_k= (k^2+k+2)/2. We consider the effect that adding an extra cut will have on the result.
Step3: Looking at the tabulated value for n and R, you will found that adding an extra chord to a circle produces an extra (n+1) regions, so that we can say that

R_{k+1}=R_k+ the extra regions added by the extra chords.
=R_k+ (k+1)
= (k^2+k+2)/2+ (k+1)
= (k^2+3k+4)/2

(k^2+3k+4)/2 is the $(k+1)^{th}$ assertion

Step4: Therefore, if the proposition is true for n=k, then it is true for n=k+1. As it is true for n=1, then it must be true for n=1+1=2. As it is true for n=2 then it must hold for n=2+1=3 and so on for all integers $n \geq 0$.

That is, by the principle of mathematical induction. P(n) is true.

Now we try to rewrite the formula in the form R=X+S, when X is an algebraic expression in n:

n	1	2	3	4	5
S	2	3	4	5	6

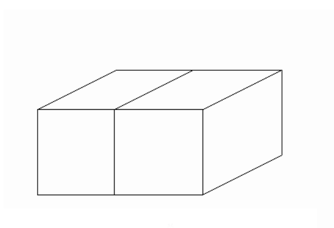
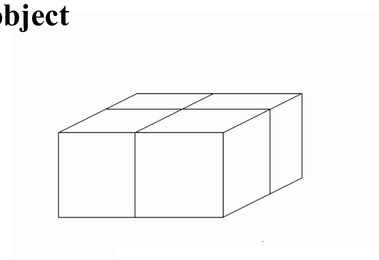
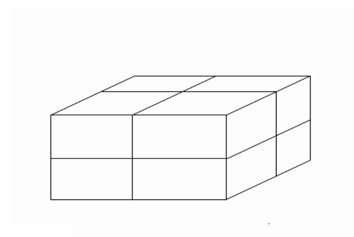
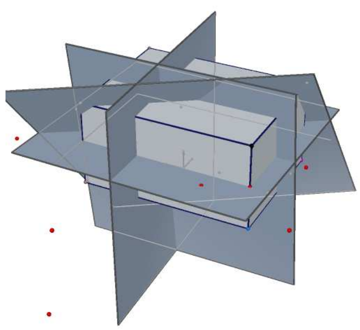
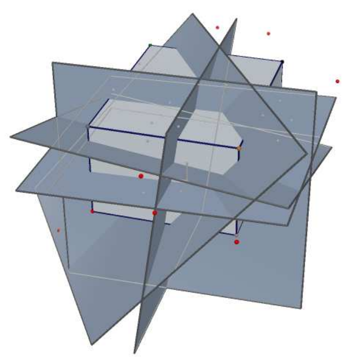
n	1	2	3	4	5
R	2	4	7	11	16

Compare these two tables, we can find out that
$R_2=R_1+S_1=2+2=4$
$R_3=R_2+S_2=4+3=7$
$R_4=R_3+S_3=7+4=11$
$R_5=R_4+S_4=11+5=16$

Thus $R=X+(n+1) \Rightarrow X=R-(n+1) \Rightarrow X=(n^2-n)/2$

Therefore, **$R=X+S=(n^2-n)/2+(n+1)$**

- **Three-dimensional object**

n	1	2	3	4	5
P	2	4	8	15	26

①Recursive rule:

$P_1=2=R_1$
$P_2=4=2+2=R_1+P_1$
$P_3=8=4+4=R_2+P_2$
$P_4=15=7+8=R_3+P_3$
$P_5=26=11+15=R_4+P_4$
Thus $P_n=P_{n-1}+R_{n-1}$

$$=P_{n-2}+R_{n-2}+\cdots\cdots+R_{n-1}$$

$=P_{n-(n-1)} + R_{n-(n-1)} + R_{n-2} + \cdots\cdots + R_{n-1}$

$=P_1+R_1+R_2+R_3+\cdots\cdots+R_{n-1}$

= 2+(1^2+1+2)/2+(2^2+2+2)/2 +⋯⋯+ (n^2+n+2)/2

= 2+(n-1)+[1*2+2*3+3*4+⋯⋯+n*(n-1)]/2

= n+1+ (n-1)*n*(n+1)]/ (3*2)
= (n^3+5n+6)/6

Therefore, the recursive rule to generate the maximum number of parts is

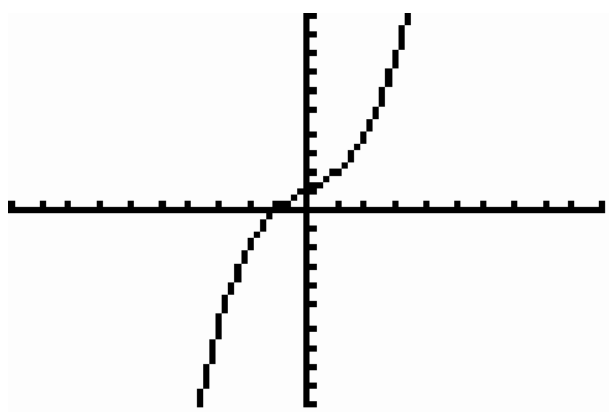
(n^3+5n+6)/6

When n=5, P_5= (5^3+5*5+6)/6=16, which corresponds to the tabulated value for n=5 above.

P_n=(n^3+5n+6)/6

②A conjecture for the relationship between the maximum number of parts(P) and the number of cuts(n).

Use TI-84 to sketch the graph related to the variables n and P suggests that the relationship between them could be cubic, and so we might assume that



$P=an^3+bn^2+cn+d$

Substituting all the values for n gives:
n=1 ⇒ 2=a+b+c+d
n=2 ⇒ 4=8a+4b+2c+d
n=3 ⇒ 8=27a+9b+3c+d
n=4 ⇒ 15=64a+16b+4c+d
n=5 ⇒ 26=125a+25b+5c+d
Solve these five equations for a, b, c, d gives
a=1/6, b=0, c=5/6, d=1

Thus $P_n = 1/6n^3 + 5/6n + 1$

When n=5, $P_5 = 1/6*5^3 + 5/6*5 + 1$, which is also corresponds to the tabulated value for n=5 above.

$P_n = 1/6n^3 + 5/6n + 1$

$P_n = (n^3+5n+6)/6$ is the same with $P_n = 1/6n^3 + 5/6n+1$

So far we have formed a conjecture that the maximum number of parts from n-cuts is given by $P_n = (n^3+5n+6)/6$.

Proof:

Let T(n) be the proposition that the maximum number of parts that can be separated from n-cuts is given by $P= (n^3+5n+6)/6$ for $n>0$

Step1: T(n) is true for n=1 as $P= (1^3+5+6)/6=2$ which is the maximum number of parts from 1 cut.

Step2: Assume that T(n) is true for a k-cuts cuboid i.e., that $P_k= (k^3+5k+6)/6$. We consider the effect that adding an extra cut will have on the result.

Step3: Looking at the tabulated value for n and P, you will found that adding an extra cut to a cuboid produces an extra $(n^2+n)/2$ parts, so that we can say that

$$\begin{aligned} P_{k+1} &= P_k + \text{the extra parts added by the extra cuts.} \\ &= P_k + (k^2+k+2)/2 \\ &= (k^3+5k+6)/6 + (k^2+k+2)/2 \\ &= (k^3+3k^2+8k+12)/6 \end{aligned}$$

Step4: $(k^3+3k^2+8k+12)/6$ is the $(k+1)^{th}$ assertion

Therefore, if the proposition is true for n=k, then it is true for n=k+1. As it is true for n=1, then it must be true for n=1+1=2. As it is true for n=2 then it must hold for n=2+1=3 and so on for all integers $n>0$.

That is, by the principle of mathematical induction. T (n) is true.

Now we try to rewrite the formula in the form P=Y+X+S where Y is an algebraic expression in n:

n	1	2	3	4	5
S	2	3	4	5	6

n	1	2	3	4	5
R	2	4	7	11	16

n	1	2	3	4	5

P	2	4	8	15	26

Compare these three tables, we can find out that
$P_2=R_1+P_1=2+2=4$
$P_3=R_2+P_2=4+4=8$
$P_4=R_3+P_3=7+8=15$
$P_5=R_4+P_4=11+15=26$
⇒P=Y+X+S=Y+ (n^2-n)/2+ (n+1)

⇒Y=P−(n^2-n)/2+ (n+1)

= (n^3+5n+6)/6-[(n^2-n)/2+ (n+1)]
= (n^3+5n+6)/6-(n^2+n+2)/2
= n^3-3n^2+2n
Therefore, **P=Y+X+S= (n^3-3n^2+2n) + (n^2-n)/2+ (n+1)**

n \ Object	One-dimensional	Two-dimensional	Three-dimensional
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26

◆ **Four-dimensional object**

Looking at the spreadsheet's value of different dimensional object for n, form a difference array, you will found that

Sequence 26 15 8 4 2 (Three -dimensional)

1^{st} differences 11 7 4 2 (Two -dimensional)

2^{st} differences 4 3 2 (One -dimensional)

3^{st} differences 1 1

As we found that all the dimensional objects separated into 2 parts when n=1, therefore when n=1, Q_1=2. Use the difference array again to find out the results of the four-dimensional.

Sequences 57 31 16 8 4 2 (Four dimensional)

1^{st} differences 26 15 8 4 2 (Three-dimensional)

2^{st} differences 11 7 4 2 (Two-dimensional)

3^{st} differences 4 3 2 (One-dimensional)

4^{st} differences 1 1

The results is shown below in a table:

n	1	2	3	4	5	6
Q	2	4	8	16	31	57

①A conjecture for the relationship between the maximum number of parts(Q) and the number of cuts(n) in a four-dimensional object. As it is a four= dimensional object, so we might assume that

Q= $an^4 + bn^3 + cn^2 + dn + e$

Substituting all the values for
n gives:

n=1 ⇒ 2=a+b+c+d+e

n=2 ⇒ 4=16a+8b+4c+2d+e

n=3 ⇒ 8=81a+27b+9c+3d+e

n=4 ⇒ 16=256a+64b+16c+4d+e

n=5 ⇒ 31=625a+125b+25c+5d+e

n=6 ⇒ 57=1296a+216b+36c+6d+e

Solve these six equations for
a, b, c, d, e gives

a=1/24, b= -1/12, c=11/24, d=7/12, e=1

Thus Q_n = $Q_n = 1/24 * n^4 - 1/12 * n^3 + 11/24 * n^2 + 7/12 * n + 1$.

When n=6, $Q_6 = 1/24*6^4 - 1/12*6^3 + 11/24*6^2 + 7/12*6 + e$ =57 which is also corresponds to the tabulated value for n=6 above.

$$Q_n = 1/24*n^4 - 1/12*n^3 + 11/24*n^2 + 7/12*n + 1$$

Proof:

Let W (n) be the proposition that the maximum number of parts that can be separated from n-cuts in a four-dimensional object is given by

$Q_n = 1/24*n^4 - 1/12*n^3 + 11/24*n^2 + 7/12*n + 1$

Step1:W(n) is true for n=1 as $Q_1 = 1/24*1^4 - 1/12*1^3 + 11/24*1^2 + 7/12*1 + 1 = 2$

which is the maximum number of parts from 1 cut.

Step2: Assume that W(n) is true for a k-cuts four-dimensional boject i.e., that $Q_k = 1/24*k^4 - 1/12*k^3 + 11/24*k^2 + 7/12*k + 1$. We consider the effect that adding an extra cut will have on the result.

Step3: Looking at the tabulated value for n and P, you will found that adding an extra cut to a four-dimensional object produces an extra (n^3+5n+6)/6 parts, so that we can say that

Q_{k+1}=Q_k+ the extra parts added by the extra cuts.

=Q_k+ (n^3+5n+6)/6

= $1/24*k^4 - 1/12*k^3 + 11/24*k^2 + 7/12*k + 1$ + (n^3+5n+6)/6

= (n^4+2n^3+11n^2+34n+48)/24

Step4: (n^4+2n^3+11n^2+34n+48)/24 is the $(k+1)^{th}$ assertion

Therefore, if the proposition is true for n=k, then it is true for n=k+1. As it is true for n=1, then it must be true for n=1+1=2. As it is true for n=2 then it must hold for n=2+1=3 and so on for all integers n>0.

That is, by the principle of mathematical induction. W (n) is true.

Now we try to rewrite the formula in the form Q=Z+Y+X+S where Z is an algebraic expression in n:

n	1	2	3	4	5
S	2	3	4	5	6

n	1	2	3	4	5

R	2	4	7	11	16

n	1	2	3	4	5
P	2	4	8	15	26

n	1	2	3	4	5	6
Q	2	4	8	16	31	57

Compare these four tables, we can find out that

$Q_2=P_1+Q_1=2+2=4$

$Q_3=P_2+Q_2=4+4=8$

$Q_4=P_3+Q_3=7+8=15$

$Q_5=P_4+Q_4=11+15=26$

$\Rightarrow$ Q=Z+Y+X+S=Z+ (n^3-3n^2+2n) + (n^2-n)/2+ (n+1)

$\Rightarrow$ Z=Q-(n^3-3n^2+2n) + (n^2-n)/2+ (n+1)

$=1/24*n^4-1/12*n^3+11/24*n^2+7/12*n+1$ －［(n^3-3n^2+2n)+(n^2-n)/2+ (n+1) (n^3+5n+6)/6-(n^2+n+2)/2］

=(n^4－26n^3+71n^2－46n)/24

=Therefore, **Q=Z+Y+X+S=(n^4－26n^3+71n^2－46n)/24 +(n^3-3n^2+2n)+(n^2-n)/2+ (n+1)**

很好！很独特的方法来找到四维的结论。

既然找到了每个维度的结果之间的关系“**DIFFERENCE ARRAY**”，相信你也能很顺利的得到五维、六维甚至更高维度空间的结论。那么能不能找到一个更一般的结论，适用于任何维度呢？好好思考下。

http://tieba.baidu.com/f?kz=280407483

Your information?

$$d_m = C_m^n + C_{m-1}^n + C_{m-2}^n + \ldots\ldots + C_4^n + C_3^n + C_2^{n+1} + 1.$$

In 2-dimension we can get R = $\frac{n(n+1)}{2}+1$

In 3-dimension we can get P = $\frac{n(n-1)(n-2)}{6}+\frac{n(n+1)}{2}+1$

In 4-dimension we can get Q=$\frac{n(n-1)(n-2)(n-3)}{24}+\frac{n(n-1)(n-2)}{6}+\frac{n(n+1)}{2}+1$

And we can rewrite the function as:

In 2D: R = $\frac{n(n+1)}{2}+1$

$= C_2^{n+1}+1$

In 3D: P= $\frac{n(n-1)(n-2)}{6}+\frac{n(n+1)}{2}+1$

$= C_3^n + C_2^{n+1}+1$

In 4D: Q=$\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}+\frac{n(n-1)(n-2)}{6}+\frac{n(n+1)}{2}+1$

$= C_4^n + C_3^n + C_2^{n+1}+1$

M

ARK： １＋２＋５＋５＋３＋１＝１７（满分 20）