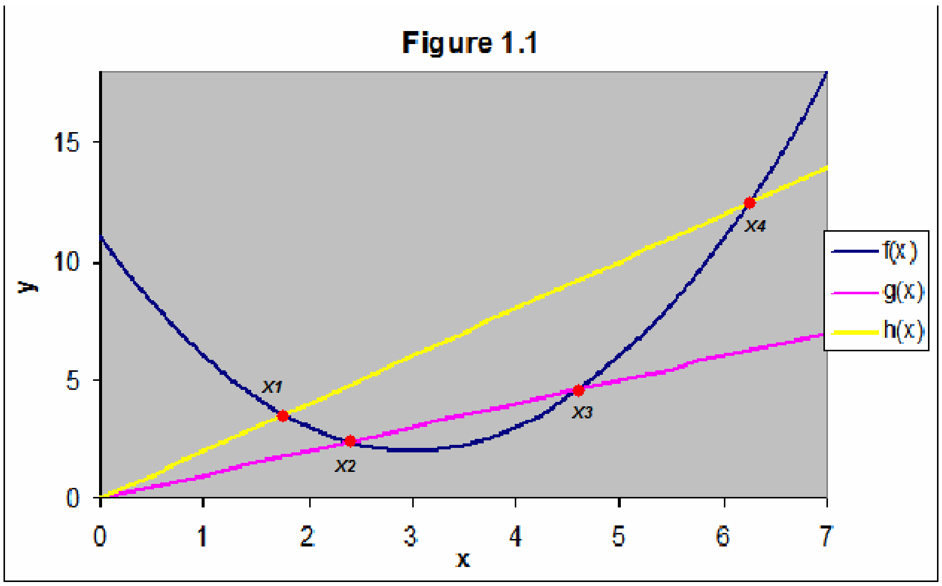
IBHL 2 Math

Parabola Investigation

In this investigation, relationships between the points of intersection of a parabola and two different lines were examined.

First, the parabola $f(x) = (x-3)^2 + 2$ and the lines $g(x) = x$ and $h(x) = 2x$ were used as an example.

As seen on the adjacent graph, the points of intersection were labeled left to right as x_1, x_2, x_3, and x_4.



Using a graphing calculator, these values were found:

x_1=1.764, x_2=2.382, x_3=4.618, and x_4=6.234.

At this point, x_1 was subtracted from x_2, and x_3 from x_4, and the resulting numbers were labeled S_L and S_R respectively:

$$2.382 - 1.764 = S_L = 0.616$$
$$6.234 - 4.618 = S_R = 1.616$$

After this, a value D was found:

$$D = |S_L - S_R|$$
$$D = |0.6\overline{6} - 1.6\overline{6}|$$
$$D = |-1|$$
$$D = 1$$

To further investigate this relationship, I followed this same process with many different parabolas and lines. These lines and their values are shown in the table below (Figure 2.1). A graph of the displayed functions is below this (Figure 2.2). An explanation for each significant situation is provided below Figure 2.2.

Figure 2.1

Situation #	$f(x)$	$g(x)$	$h(x)$	D
2	$(x-7)^2+4$	x	$2x$	1
3	$\frac{1}{4}(x-8)^2+9$	x	$2x$	4
4	$\frac{1}{2}(x-5)^2+1$	x	$2x$	2
5	$5(x-3)^2+2$	x	$2x$	$\frac{1}{5}$
6	$-3(x-7)^2+26$	x	$2x$	$\frac{1}{3}$
7	$\sqrt{2}(x-5)^2+4$	x	$2x$	$\frac{1}{\sqrt{2}}$
8	$\sqrt{2}(x+4)^2-11$	x	$2x$	$\frac{1}{\sqrt{2}}$
9	$-8(x+5)^2-5$	x	$2x$	$\frac{1}{8}$
10	$-\frac{3}{2}(x)^2+\frac{5}{2}$	x	$2x$	$\frac{2}{3}$

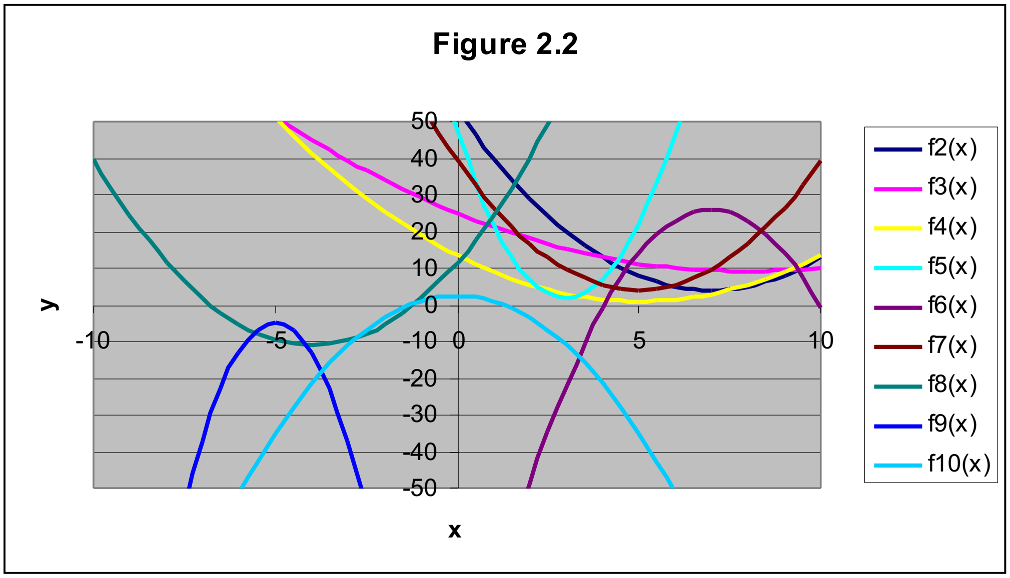


Figure 2.2

Situation 3: This situation caused me to hypothesize the conjecture that $D = A^{-1}$, where A is the A value from the equation $f(x) = A(x-B)^2 + C$. In addition, $g(x)$ was tangent to $f(x)$ at the point (10,10). This meant that x_2 and x_3 were the same (10), and showed that the conjecture held true for tangents.

Situation 6: A concave down parabola in the 1st quadrant still holds the conjecture, provided that the conjecture is changed to $D = \left|A^{-1}\right|$

Situation 7: Irrational A values work for the conjecture.

Situation 8: A concave up parabola in the third quadrant works.

Situation 9: An $f(x)$ that is concave down in the third quadrant works.

Situation 10: Intersections in both 1st and 3rd quadrants work as long as the intersections of one line are x_2 and x_3, and the intersections of the other line are x_1 and x_4.

After situation 10, I proved my conjecture that when the lines g(x)=x and h(x)=2x intersect the parabola $f(x) = A(x-B)^2 + C$, the value of D is $\frac{1}{A}$

$$f(x) = A(x-B)^2 + C = Ax^2 - 2ABx + AB^2 + C$$

$$f(x) = g(x)$$

$$\therefore Ax^2 - 2ABx + AB^2 + C = x$$

$$\therefore Ax^2 + x(-2AB - 1) + AB^2 + C = 0$$

Using the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, I found the intersections of lines $f(x)$ and $g(x)$: x_2 and x_3.

$$a = A$$
$$b = (-2AB - 1)$$
$$c = (AB^2 + C)$$
$$\therefore \frac{2AB + 1 \pm \sqrt{(-2AB - 1)^2 - 4(A)(AB^2 + C)}}{2A}$$
$$\therefore \frac{2AB + 1 \pm \sqrt{4A^2B^2 + 4AB + 1 - 4A^2B^2 - 4AC)}}{2A}$$
$$\therefore \frac{2AB + 1 \pm \sqrt{4AB + 1 - 4AC)}}{2A}$$
$$x_2 = \frac{2AB + 1 - \sqrt{4AB + 1 - 4AC)}}{2A}$$
$$x_3 = \frac{2AB + 1 + \sqrt{4AB + 1 - 4AC)}}{2A}$$

Using the same method, x_1 and x_4 could also be found.

$$a = A$$
$$b = (-2AB - 2)$$
$$c = (AB^2 + C)$$
$$\therefore \frac{2AB + 2 \pm \sqrt{(-2AB - 2)^2 - 4(A)(AB^2 + C)}}{2A}$$
$$\therefore \frac{2AB + 2 \pm \sqrt{4A^2B^2 + 8AB + 4 - 4A^2B^2 - 4AC)}}{2A}$$
$$\therefore \frac{2AB + 2 \pm \sqrt{8AB + 4 - 4AC)}}{2A}$$
$$x_1 = \frac{2AB + 2 - \sqrt{8AB + 4 - 4AC)}}{2A}$$
$$x_4 = \frac{2AB + 2 + \sqrt{8AB + 4 - 4AC)}}{2A}$$

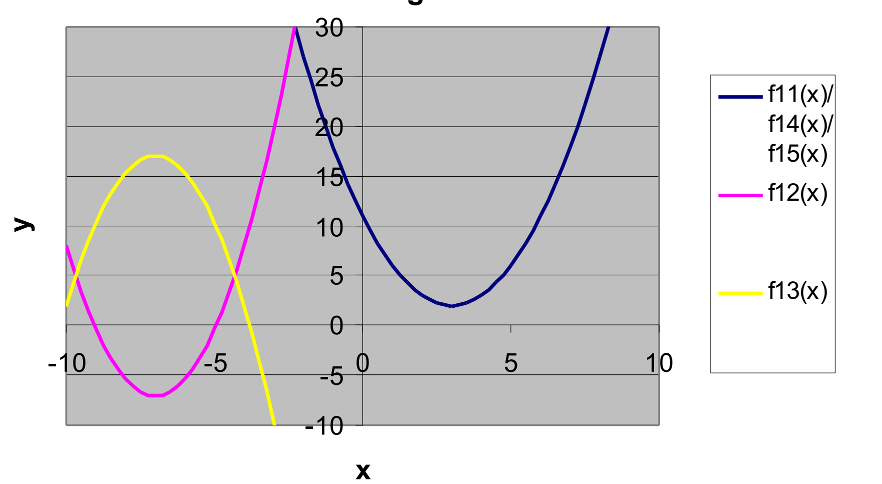
$$\therefore S_L = \frac{2AB + 1 - \sqrt{4AB + 1 - 4AC)} - 2AB - 2 + \sqrt{8AB + 4 - 4AC)}}{2A}$$
$$\therefore \frac{-1 - \sqrt{4AB + 1 - 4AC)} + \sqrt{8AB + 4 - 4AC)}}{2A}$$
$$\therefore S_R = \frac{2AB + 2 + \sqrt{8AB + 4 - 4AC)} - 2AB - 1 - \sqrt{4AB + 1 - 4AC)}}{2A}$$
$$\therefore \frac{1 + \sqrt{8AB + 4 - 4AC)} - \sqrt{4AB + 1 - 4AC)}}{2A}$$
$$\therefore D = \left| \frac{-1 - \sqrt{4AB + 1 - 4AC)} + \sqrt{8AB + 4 - 4AC)} - 1 - \sqrt{8AB + 4 - 4AC)} + \sqrt{4AB + 1 - 4AC)}}{2A} \right|$$
$$\therefore \left| \frac{-2}{2A} \right|$$
$$\therefore D = \left| \frac{-1}{A} \right|$$

Here, $g(x)$ and $h(x)$ were changed to see if this changed the conjecture. The situations tested appear algebraically below in Figure 3.1, and graphically in Figure 3.2

Figure 3.1

Situation #	$f(x)$	$g(x)$	$h(x)$	D
11	$(x-3)^2+2$	x	$3x$	2
12	$-\frac{5}{3}(x-7)^2-7$	$2x$	$4x$	1.2
13	$-\frac{5}{3}(x-7)^2+17$	$-2x$	$4x$	3.6
14	$(x-3)^2+2$	$x+1$	$2x$	1
15	$(x-3)^2+2$	$x+1$	$2x+5$	1

Figure 3.2



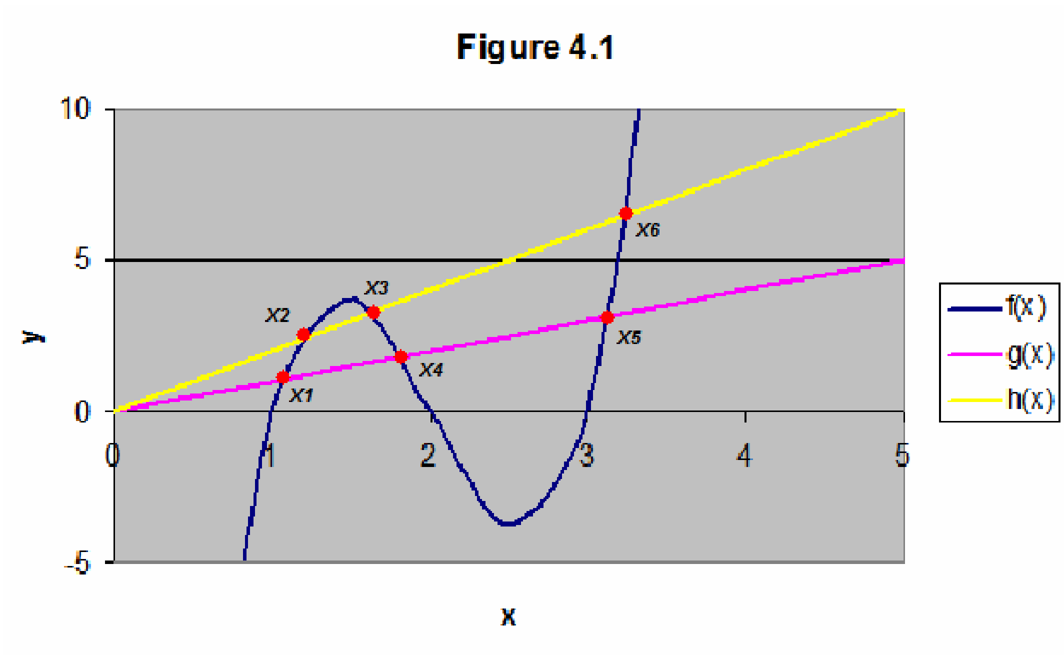
Situation 11: Here, I changed my conjecture to tentatively be $D=\left|\frac{n-m}{A}\right|$, where m is from $g(x)=mx$ and n is from $h(x)=nx$.

Situation 14: Here I attempted to determine whether the b value in $g(x)=mx+b$ affected the D value. It did not, most likely because the part of the intersection point this problem is concerned with is the x value, and with parabolas, the x value increases proportionally on both sides of the axis of symmetry.

After trying all of these parabolas, I again proved my conjecture, the only difference being that now $g(x)=mx$ and $h(x)=nx$.

$$a = A$$
$$b = (-2AB - m)$$
$$c = (AB^2 + C)$$
$$\therefore \frac{2AB + m \pm \sqrt{(-2AB - m)^2 - 4(A)(AB^2 + C)}}{2A}$$
$$\therefore \frac{2AB + m \pm \sqrt{4A^2B^2 + 4ABm + m^2 - 4A^2B^2 - 4AC)}}{2A}$$
$$\therefore \frac{2AB + m \pm \sqrt{4ABm - 4AC + m^2)}}{2A}$$
$$x_1 = \frac{2AB + n - \sqrt{4ABn - 4AC + n^2)}}{2A}$$
$$x_2 = \frac{2AB + m - \sqrt{4ABm - 4AC + m^2)}}{2A}$$
$$x_3 = \frac{2AB + m + \sqrt{4ABm - 4AC + m^2)}}{2A}$$
$$x_4 = \frac{2AB + n + \sqrt{4ABn - 4AC + n^2)}}{2A}$$
$$\therefore S_L = \frac{2AB + m - \sqrt{4ABm - 4AC + m^2)} - 2AB - n + \sqrt{4ABn - 4AC + n^2)}}{2A}$$
$$\therefore = \frac{m - \sqrt{4ABm - 4AC + m^2)} - n + \sqrt{4ABn - 4AC + n^2)}}{2A}$$
$$\therefore S_R = \frac{2AB + n + \sqrt{4ABn - 4AC + n^2)} - 2AB - m - \sqrt{4ABm - 4AC + m^2)}}{2A}$$
$$\therefore = \frac{n + \sqrt{4ABn - 4AC + n^2)} - m - \sqrt{4ABm - 4AC + m^2)}}{2A}$$
$$\therefore D = \left| \frac{m - \sqrt{4ABm - 4AC + m^2)} - n + \sqrt{4ABn - 4AC + n^2)} - n - \sqrt{4ABn - 4AC + n^2)} + m + \sqrt{4ABm - 4AC + m^2)}}{2A} \right|$$
$$\therefore = \left| \frac{2m - 2n}{2A} \right|$$
$$\therefore D = \left| \frac{m - n}{A} \right|$$

After proving this conjecture, I investigated similar situations with cubic polynomials. Shown below is the graph of $f(x) = 10(x-1)(x-2)(x-3)$, $g(x) = x$, and $h(x) = 2x$ (Figure 4.1).



The intersections, similar to the quadratic examples, are labeled as x_1, x_2, x_3, x_4, x_5, and x_6 from left to right as shown.

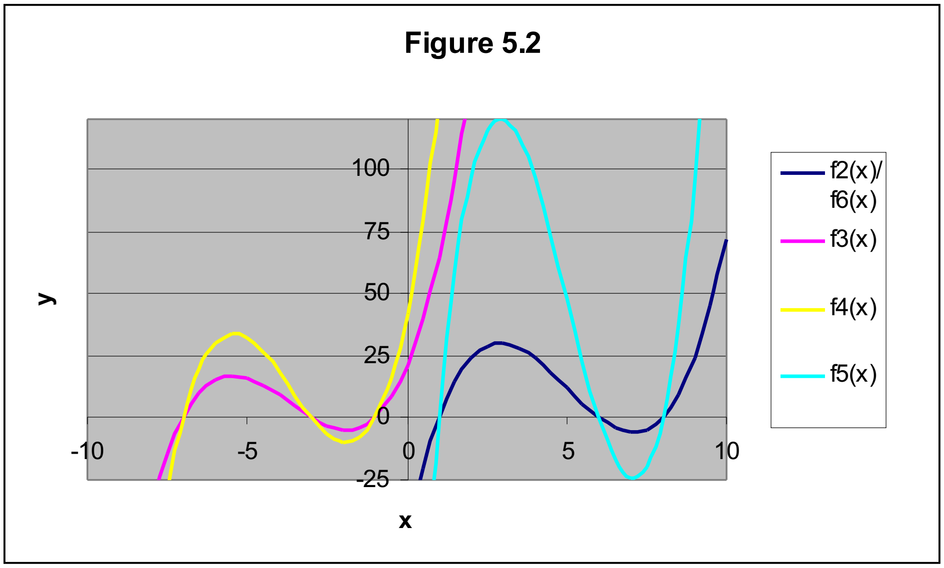
In this case, I again used my calculator to determine the values of these intersections. Once these were calculated, the values of $S_L = x_2 - x_1$, $S_M = x_4 - x_3$, and $S_R = x_6 - x_5$ were determined:

$$S_L = 1.144 - 1.058 = 0.086$$
$$S_M = 1.812 - 1.621 = 0.191$$
$$S_R = 3.235 - 3.130 = 0.105$$

After rearranging these values I found that $S_M - S_L - S_R = 0$, and then tested this conjecture on other cubic functions, shown below algebraically in Figure 5.1, and graphically in Figure 5.2

Figure 5.1

Example #	$f(x)$	$g(x)$	$h(x)$	S_L	S_M	S_R
2	$(x-1)(x-6)(x-8)$	x	$2x$	0.033	0.376	0.344
3	$(x+3)(x+7)(x+1)$	x	$2x$	-0.24	-0.39	-0.149
4	$2(x+3)(x+7)(x+1)$	x	$2x$	-0.131	-0.184	-0.053
5	$4(x-1)(x-6)(x-8)$	x	$2x$	0.008	0.126	0.119
6	$(x-1)(x-6)(x-8)$	x	$3x$	0.068	0.697	0.63



Example 3 = Cubics with negative roots work with the conjecture.

Example 3 = The S_M, S_L, and S_R values of this example were about half of those in example 1, leading me to change my conjecture to $\frac{1}{A}(S_M - S_L - S_R) = 0$

Example 5 = Here, as in the quadratic, I changed the $h(x)$ equation to $3x$ to observe its affect on the S_M, S_L, and S_R values. When this change showed a change in the S values I formed the conjecture that $\frac{m-n}{A}(S_M - S_L - S_R) = 0$.