International Baccalaureate
Math Methods SL Portfolio 1:

Logarithms

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The intent of this portfolio is to explore, investigate, and model the patterns found in logarithmic functions.

In considering the first logarithm in the first sequence:

$\log_2 8$

It is important to know what it represents.
By definition, a logarithm is the inverse of an exponential function.
Therefore, when:

$2^x = 8$
$x = 3$

The next expression in the series is:

$\log_4 8$

By definition:

$4^x = 8$

This expression is more difficult to solve as easily.
However, if 8 is seen instead as:

2^3

The expression becomes easier to solve

$4^x = 2^{2x}$
$2^{2x} = 2^3$
$2x = 3$
$x = \frac{3}{2}$

There is a pattern in the bases of each consecutive expression.
In the first set, the base is defined as:

2^n

where n is the nth term in the sequence.

In considering the whole sequence:

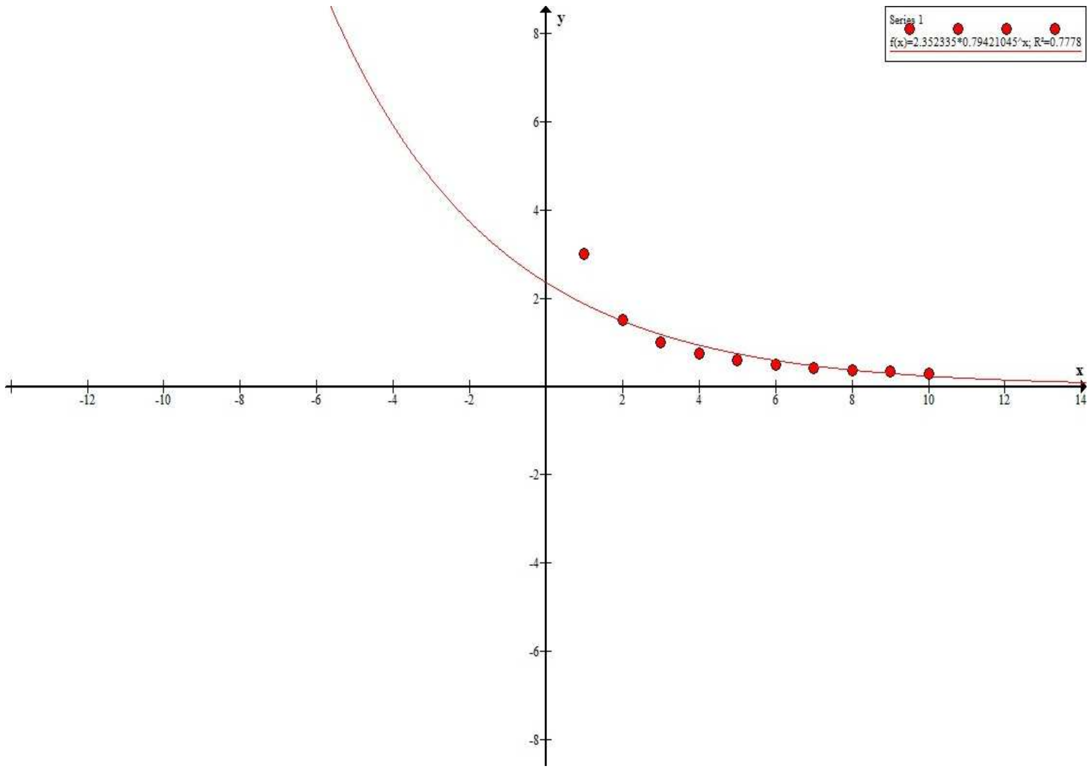
Sequence 1:

Term 1	Term 2	Term 3	Term 4	Term 5
$\log_2 8$	$\log_4 8$	$\log_8 8$	$\log_{16} 8$	$\log_{32} 8$

So, in term 5, the base is 2^5
Therefore, in terms 6 and 7 of all sequences, the value *n* must be equal to 6 and 7 respectively

Term 6	Term 7
$\log_{64} 8$	$\log_{128} 8$

The graph below represents the resulting values in sequence 1 as the exponent of the base increases.



Notice, the values of y never actually reach 0 because as the base increases, the smaller and closer to 1 the result must be in order to fit the expression.

The same can be done with the other sequences:

Next two terms of the sequence

$\log_3 81$	$\log_9 81$	$\log_{27} 81$	$\log_{81} 81$	$\log_{243} 81$	$\log_{729} 81$
$\log_5 25$	$\log_{25} 25$	$\log_{125} 25$	$\log_{625} 25$	$\log_{3125} 25$	$\log_{1556275} 25$
$\log_m m^k$	$\log_{m^2} m^k$	$\log_{m^3} m^k$	$\log_{m^4} m^k$	$\log_{m^5} m^k$	$\log_{m^6} m^k$

It is shown by this final sequence that the exponent of the base is increasing with direct correlation to the number of the term.
The constant *k,* however, stays the same in all sequences.

In looking back once more to the first sequence:

Sequence 1

Term 1	Term 2	Term 3	Term 4	Term 5
$\log_2 8$	$\log_4 8$	$\log_8 8$	$\log_{16} 8$	$\log_{32} 8$

The results show a pattern in relation to some of the different terms in the original statement:

$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$

The top value, 3, represents the constant *k*.
The bottom value, increasing by 1 after each term of the sequence, represents the term number of the sequence.

This can by proved by finding the results of the final sequence, in terms of *m:*

$\log_m m^k$	$\log_{m^2} m^k$	$\log_{m^3} m^k$	$\log_{m^4} m^k$	$\log_{m^5} m^k$	$\log_{m^6} m^k$

Results

$\frac{k}{1}$	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{4}$	$\frac{k}{5}$	$\frac{k}{6}$

The numerator stays at *k,* and since the base of the exponent is represented by *n*, the denominator is also represented by *n*.

Therefore, the n^{th}term of the sequence in terms of p/q is $\frac{k}{n}$
It is important to look at finding the solution to the answer in a different way.
The change of base rule is generally a rule that puts the logarithmic statement into one that is expressed in base 10, so it is easily calculatable on the calculator.

The change of base rule is this:
For $\log_a x = b$

$$b = \frac{\log x}{\log a}$$

This can easily be proved by considering the solution by definition of a logarithm.
$a^b = x$

Now, calculate the following, giving your answers in the form p/q

Sequence 2a	$\log_4 64$	$\log_8 64$	$\log_{32} 64$
Results	$3/1$	2	1.2

Sequence 2b	$\log_7 49$	$\log_{49} 49$	$\log_{343} 49$
Results	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$

Sequence 2c	$\log_{\frac{1}{5}}125$	$\log_{\frac{1}{125}}125$	$\log_{\frac{1}{625}}125$
Results	$\frac{3}{-1}$	$\frac{3}{-3}$	$\frac{3}{-4}$

Sequence 2d	$\log_8 512$	$\log_2 512$	$\log_{16} 512$
Results	$\frac{9}{3}$	$\frac{9}{1}$	$\frac{9}{4}$

Figuring out the third expression in the sequence is a matter of adding together the exponents of the base.

In the first example, the bases have a similar root-- namely, 2.

Sequence 2a	$\log_4 64$	$\log_8 64$	$\log_{32} 64$

Think of the bases not as separate values, but instead as exponents of 2.

Sequence 2a(Revised)	$\log_{2^2} 64$	$\log_{2^3} 64$	$\log_{2^5} 64$

From the results above, it is easy to suspect that the third term of the sequence comes from an addition of the exponents of the bases in the two previous terms:
2+3=5

The same can be done with the next three sequences:

Sequence 2b	$\log_7 49$	$\log_{49} 49$	$\log_{343} 49$
Sequence 2b(Revised)	$\log_{7^1} 49$	$\log_{7^2} 49$	$\log_{7^3} 49$

From the results above, it is seen that the exponent of the third term is the sum of the two base exponents in the terms preceding it.
1+2=3

Sequence 2c	$\log_{\frac{1}{5}}125$	$\log_{\frac{1}{125}}125$	$\log_{\frac{1}{625}}125$
Sequence 2c(Revised)	$\log_{5^{-1}} 125$	$\log_{5^{-3}} 125$	$\log_{5^{-4}} 125$

(-1)+(-3)=(-4)

Sequence 2d	$\log_8 512$	$\log_2 512$	$\log_{16} 512$
Sequence 2d(Revised	$\log_{2^3} 512$	$\log_{2^1} 512$	$\log_{2^4} 512$

3+1=4

Let $\log_a x = c$and $\log_b x = d$. Find the general statement that expresses$\log_{ab} x$, in terms of *c* and *d*.

In order to do put these two expressions together, one must first find a common base. This can be done by taking the inverse of these two equations and make x as the base of the two.
In order to do this, however, it must be proven that $\log_a x$is equal to $\frac{1}{\log_x a}$
This can be done by using trial and error.
If *a = 2* and *x = 8:*
It has already been solved that $\log_2 8 = 3$

If $\frac{1}{\log_x a}$is also equal to 3, $\log_x a$must be equal to $\frac{1}{3}$

Therefore, $\log_8 2$must also equal $\frac{1}{3}$
It is best to rewrite $\log_8 2$as $\log_{2^3} 2$
By rules of logarithms: $(2^3)^x = 2$
Since the two have the same base, it can be eliminated.
$3\mathrm{x} = 1$
$$x = \frac{1}{3}$$
Plugging in, $\frac{1}{1/3} = 3$
Thus:
$\frac{1}{\log_x a} = \log_a x$ Equation 1

Using this new formula, it is now possible to group the two initial equations

$\log_a x = c$and $\log_b x = d$can now be written as:

$\frac{1}{\log_x a} = \frac{1}{c}$and $\frac{1}{\log_x b} = \frac{1}{d}$

$$\frac{1}{\log_x a} + \frac{1}{\log_x b} = \frac{1}{c} + \frac{1}{d}$$

Because the two bases in the logarithmic equations are the same, it is possible to join them.

$$\frac{1}{\log_x ab} = \frac{1}{c} + \frac{1}{d}$$

Using Equation 1, the base in the left side equation can now be changed back to ab:
$$\frac{1}{\log_x ab} = \log_{ab} x$$

The right side of the equation is now in terms of *c* and *d;* however, both variables are in the denominator.

$$\frac{1}{c} + \frac{1}{d}$$
By multiplying by the common denominator *(c+d)* it is possible to group these two terms together

General statement

$$(\frac{1}{c}+\frac{1}{d})(\frac{c+d}{c+d})$$

$$\frac{d}{cd}+\frac{c}{cd}$$

Now that there is a common denominator, it is possible to group the terms together to get the general statement.

$\frac{c+d}{cd}$, and because we had to flip the left side of the equation, we must do the same on the right side of the equation:

$\frac{cd}{c+d}$

Proof using Trial and Error:

Let $\log_a x = c$ and $\log_b x = d$

a = 4
b = 16
x = 4

$$\log_4 4 = 1$$
$$\log_{16} 4 = \frac{1}{2}$$

$\log_{ab} x$ is:

$\frac{(\frac{1}{2})}{1+\frac{1}{2}} = \frac{1}{3}$

$\log_{4*16} 4 = \frac{1}{3}$

The general statement for the two equations can also be solved by using the change of base formula:
$\log_a x = \frac{logx}{loga}$ and $\log_b x = \frac{logx}{logb}$

By taking the inverse of both—proved in the methods above—it is possible to find a common base and put the two equations together.

$\frac{1}{\log_a x} = \frac{loga}{logx}$ And $\frac{1}{\log_b x} = \frac{logb}{logx}$

Adding the two together:

$$\frac{loga}{logx} = \frac{logb}{logx} = \frac{logab}{logx}$$

Then, by taking the inverse once more, the result for the general statement can be reached:

$\frac{logx}{logab} = \frac{1}{c} + \frac{1}{d} = \frac{cd}{c+d}$

When solving for the general statement, and testing for values of *a, b,* and *x,* using the change of base formula caused some problems:

$$\log_a x = \frac{logx}{loga}$$

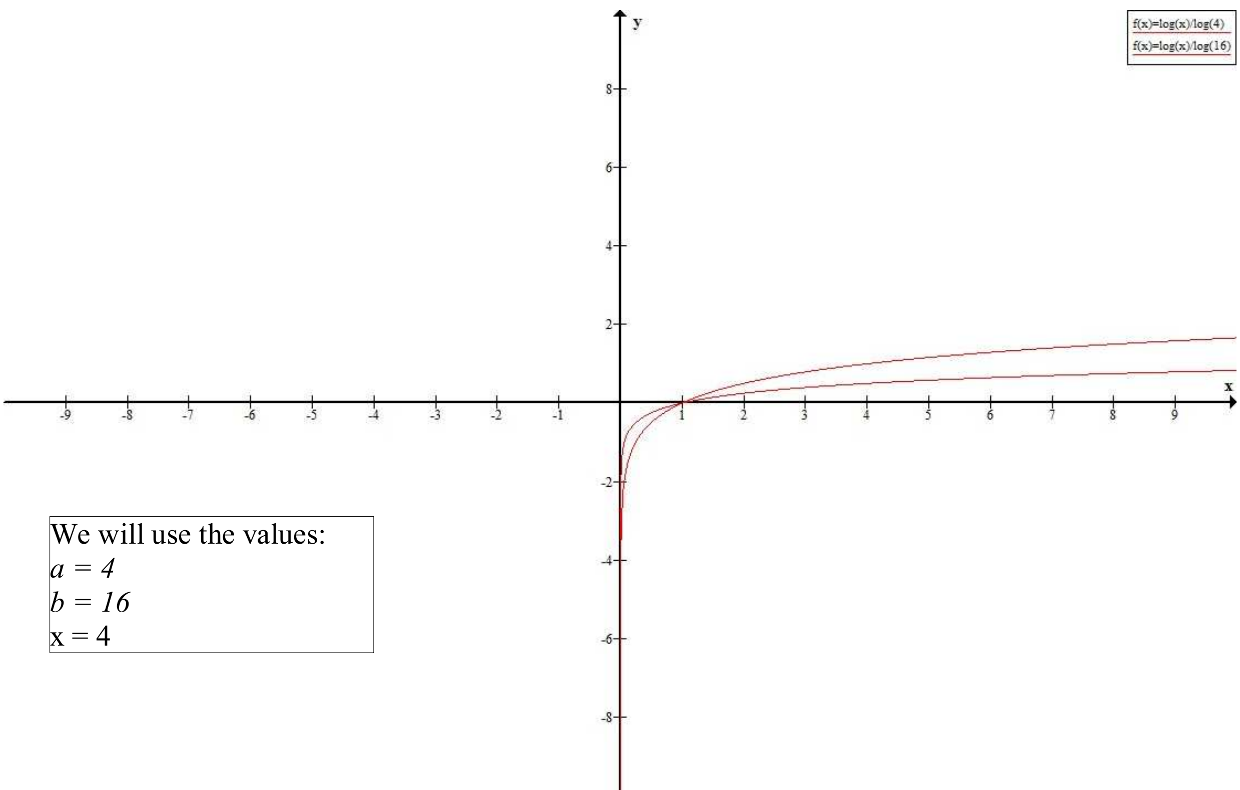
By this thought, both *x* and *a* cannot be negative, because 10 to the power of any real number equal to a negative number.

The same can be said for b:

$$\log_b x = \frac{logx}{logb}$$

x and *b* cannot be negative.

In graphing the two equations, the asymptotes and limits can be seen as to where the values do not exist.



We will use the values:
a = 4
b = 16
x = 4

It is seen in the graph that the x values of the graph are limited to positive values over 0.
in replacing *x* with 0, it is seen that the value cannot exist, because no integer *a or b,* (*a or b* > 0) to the power of another integer (over 0) can equal a value of 0, but can approach it very closely.

The result for the general statement arrives from different methods. In using the change of base formula as well as using a proven inverse rule, adding together two logarithms of different bases is possible.