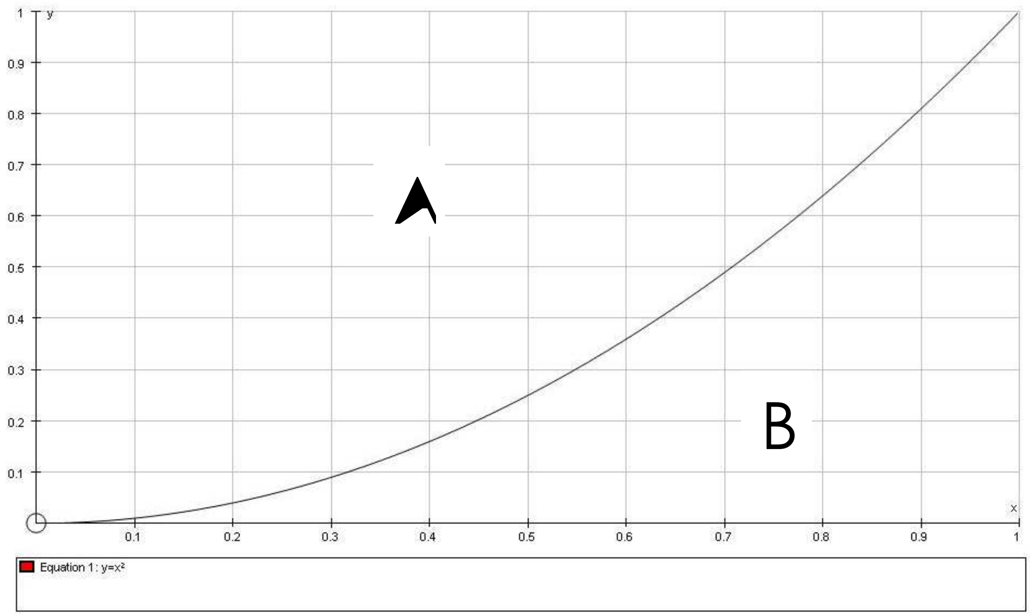
Investigating Ratios of Areas and Volumes

The aim of this portfolio is to investigate the ratio of areas form when $y = x^n$ is graphed between two arbitrary parameters $x = a$ and $x = b$ such that $a < b$.



1. Given the function $y = x^2$, consider the region formed by this function from $x = 0$ to $x = 1$ and the x-axis. This area is labeled B. The region from $y = 0$ and $y = 1$ and the y-axis is labeled A.

   Finding the ratio of area A: area B:

$$Area\ B = \int_0^1 x^2\,dx$$

$$Area\ B = \left[\frac{x^3}{3}\right]_0^1$$

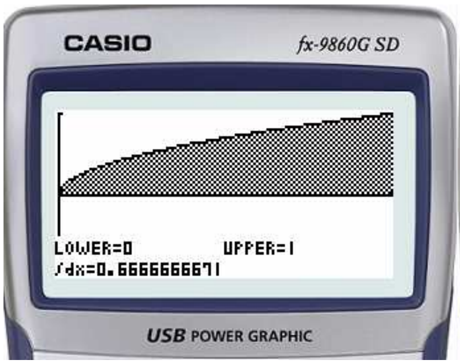
$$Area\ B = \frac{1}{3}$$

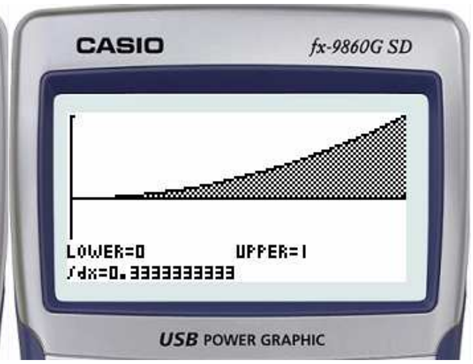
$$Area\ A = \int_0^1 y^{\frac{1}{2}}\, dy$$

$$Area\ A = \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

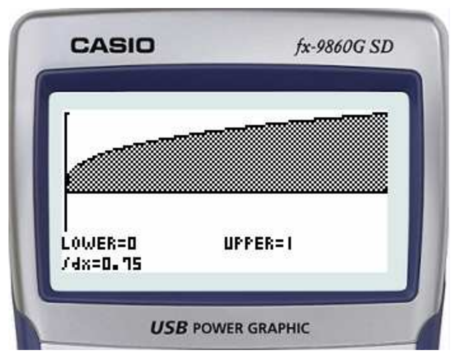
$$Area\ A = \frac{2}{3}$$

∴ The ratio of area A: area B is 2:1.





Calculate the ratio of the areas for other functions of the type $y = x^n, n \in \mathbb{Z}^+$ between $x = 0$ and $x = 1$.

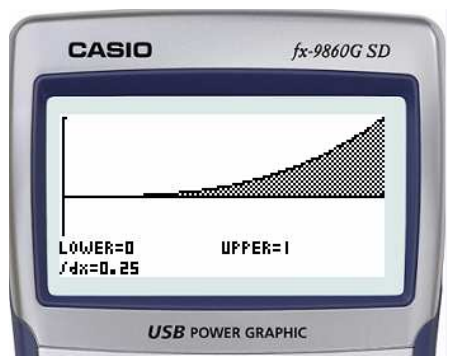


Let $n = 3$,

$y = x^3$

$x = y^{\frac{1}{3}}$

Finding the ratio of area A: area B:



$$Area\ B = \int_0^1 x^3\, dx$$

$$Area\ B = \left[\frac{x^4}{4}\right]_0^1$$

$$Area\ B = \frac{1}{4}$$

$$Area\ A = \int_0^1 y^{\frac{1}{3}}\, dy$$

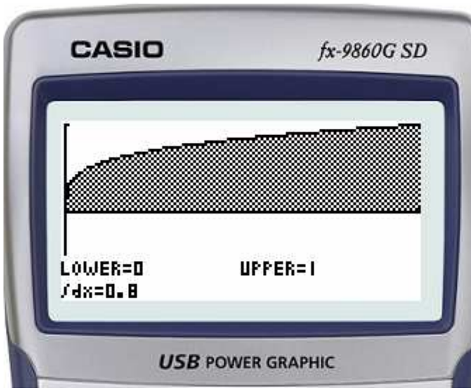
$$Area\ A = \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^1$$

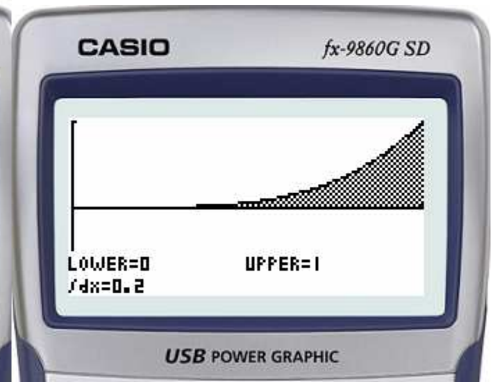
$$Area\ A = \frac{3}{4}$$

∴ The ratio of area A: area B is 3:1.

Let $n = 4$,
$y = x^4$
$x = y^{\frac{1}{4}}$

Finding the ratio of area A: area B:

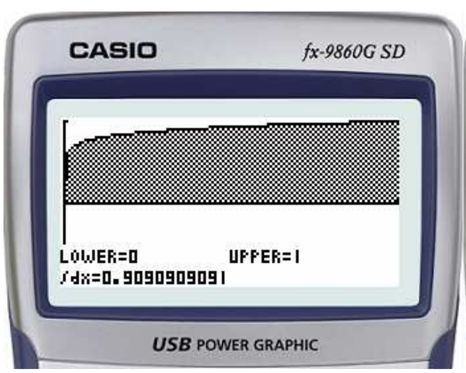


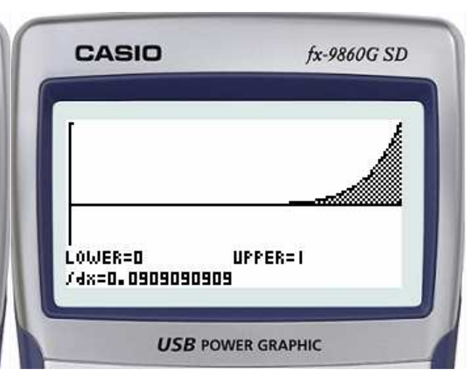


∴ The ratio of area A: area B is 4:1.

Let $n = 10$,
$y = x^{10}$
$x = y^{\frac{1}{10}}$

Finding the ratio of area A: area B:





∴ The ratio of area A: area B is 10:1.

From the findings above, it is evident that when $= x^n$, the ratio of area A: area B is always n: 1 when n is a positive integer more than 0.

To further investigate my conjecture, test the conjecture for other subsets of the real numbers:

Fractions:

Let $n = \frac{1}{2}$,

$y = x^{\frac{1}{2}}$

$x = y^2$

Finding the ratio of area A: area B:

$$Area\,B = \int_0^1 x^{\frac{1}{2}}\,dx$$

$$Area\,B = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$Area\,B = \frac{2}{3}$$

$$Area\,A = \int_0^1 y^2\,dy$$

$$Area\,A = \left[\frac{y^3}{3}\right]_0^1$$

$$Area\,A = \frac{1}{3}$$

∴ The ratio of area A: area B is $\frac{1}{2}$:1, which means that the conjecture is true to fraction values of n.

Negative Integers:

Let $n = -1$,

$y = x^{-1}$

$x = y^{-1}$

Finding the ratio of area A: area B:

$$Area\ B = \int_0^1 x^{-1}\,dx$$

$$Area\ B = \left[\frac{x^0}{0}\right]_0^1$$

$$Area\ B = No\ solution.$$

$$Area\ A = \int_0^1 y^{-1}\ dy$$

$$Area\ A = \left[\frac{y^0}{0}\right]_0^1$$

$$Area\ A = No\ solution.$$

∴ The conjecture is not true to negative integers.

Irrational Numbers:
Let $n = \pi$,

$y = x^{\pi}$

$x = y^{\frac{1}{\pi}}$

Finding the ratio of area A: area B:

$$Area\ B = \int_0^1 x^{\pi}\,dx$$

$$Area\ B = \left[\frac{x^{\pi+1}}{\pi+1}\right]_0^1$$

$$Area\ B = \frac{1}{\pi+1}$$

$$Area\ A = \int_0^1 y^{\frac{1}{\pi}}\ dy$$

$$Area\ A = \left[\frac{y^{\frac{1}{\pi}+1}}{\frac{1}{\pi}+1}\right]_0^1$$

$$Area\ A = \frac{1}{\frac{1}{\pi}+1}$$

∴ The ratio of area A: area B is π:1. This means the conjecture is true to irrational values of n.

2. This conjecture is further tested for areas not only limited between $x = 0$ and $x = 1$.

$y = x^2$, for region formed by this function from $x = 0$ to $x = 2$ and the x-axis. This area is labeled B. The region from $y = 0$ and $y = 4$ and the y-axis is labeled A.

$x = y^{\frac{1}{2}}$

Finding the ratio of area A: area B:

$$Area\ B = \int_0^2 x^2\,dx$$

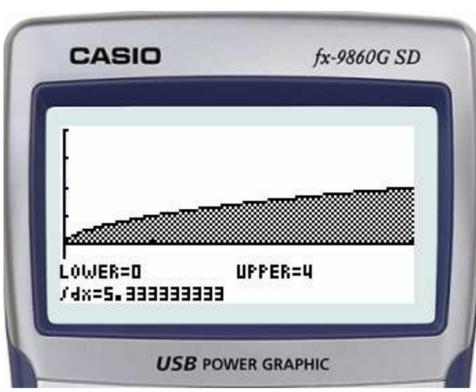
$$Area\ B = \left[\frac{x^3}{3}\right]_0^2$$

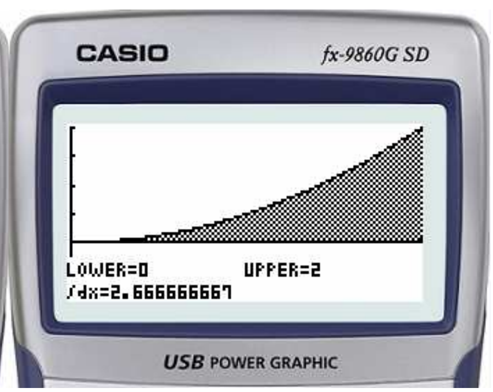
$$Area\ B = \frac{8}{3}$$

$$Area\ A = \int_0^4 y^{\frac{1}{2}}\,dy$$

$$Area\ A = \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$Area\ A = \frac{16}{3}$$





∴ The ratio of area A: area B is 2:1.

$y = x^2$, for region formed by this function from $x = 1$ to $x = 2$ and the x-axis. This area is labeled B. The region from $y = 1$ and $y = 4$ and the y-axis is labeled **A**.

$x = y^{\frac{1}{2}}$

Finding the ratio of area **A**: area B:

$$Area\,B = \int_1^2 x^2\,dx$$

$$Area\,B = \left[\frac{x^3}{3}\right]_1^2$$

$$Area\,B = \frac{7}{3}$$

$$Area\,A = \int_1^4 y^{\frac{1}{2}}\,dy$$

$$Area\,A = \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$Area\,A = \frac{14}{3}$$

∴ The ratio of area **A**: area B is 2:1.

$y = x^3$, for region formed by this function from $x = 0$ to $x = 3$ and the x-axis. This area is labeled B. The region from $y = 0$ and $y = 27$ and the y-axis is labeled **A**.

$x = y^{\frac{1}{3}}$

Finding the ratio of area **A**: area B:

$$Area\,B = \int_0^3 x^3\,dx$$

$$Area\,B = \left[\frac{x^4}{4}\right]_0^3$$

$$Area\,B = \frac{81}{4}$$

$$Area\,A = \int_0^{27} y^{\frac{1}{3}}\,dy$$

$$Area\,A = \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^{27}$$

$$Area\,A = \frac{243}{4}$$

∴ The ratio of area A: area B is 3:1.

From the above findings, it can therefore be concluded that my conjecture is not only limited to areas between $x = 0$ and $x = 1$, but holds true to other values of x as well - from $x = a$ and $x = b$ such that $a < b$.

3. Considering if my conjecture is true for the general case $y = x^n$ from $x = a$ and $x = b$ and for the regions:

   Area A: $y = x^n$, $y = a^n$, $y = b^n$ and the y-axis
   Area B: $y = x^n$, $x = a$, $x = b$ and the x-axis.

   In order to determine this, trial and error substitution is used.

   Take into account the function $y = x^2$, and the region formed by this function from $x = 0$ to $x = 1$ and the x-axis (Area A). To find the area formed on the y-axis (Area B), substitute $x = a$ and $x = b$ values as 0 and 1, into the equations $y = a^2$ and $y = b^2$, making $y = 0$ and $y = 1$. The ratio formed between area A and area B is as calculated above in Q1, 2:1.

   Then take into account the function $y = x^4$, and the region formed by this function from $x = 1$ to $x = 2$ and the x-axis (Area A). To find the area formed on the y-axis (Area B), substitute $x = a$ and $x = b$ values as 1 and 2, into the equations $y = a^4$ and $y = b^4$, making $y = 1$ and $y = 16$.
   Calculating the ratios of areas A:B:

$$Area\,B = \int_1^2 x^4\,dx$$

$$Area\,B = \left[\frac{x^5}{5}\right]_1^2$$

$$Area\,B = \frac{31}{5}$$

$$Area\ A = \int_1^{16} y^{\frac{1}{4}}\, dy$$

$$Area\ A = \left[\frac{y^{\frac{5}{4}}}{\frac{5}{4}}\right]_1^{16}$$

$$Area\ A = \frac{124}{5}$$

∴ The ratio of area A: area B is 4:1.

Next take into account the function $y = x^3$, and the region formed by this function from $x = 1$ to $x = 2$ and the x-axis (Area A). To find the area formed on the y-axis (Area B), substitute $x = a$ and $x = b$ values as 1 and 2, into the equations $y = a^3$ and $y = b^3$, making $y = 1$ and $y = 8$.

Calculating the ratios of areas A:B:

$$Area\ B = \int_1^{2} x^3\, dx$$

$$Area\ B = \left[\frac{x^4}{4}\right]_1^{2}$$

$$Area\ B = \frac{15}{4}$$

$$Area\ A = \int_1^{8} y^{\frac{1}{3}}\, dy$$

$$Area\ A = \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^{8}$$

$$Area\ A = \frac{45}{4}$$

∴ The ratio of area A: area B is 3:1.

From the above three trials we can successfully conclude that my conjecture is true for the general case $y = x^n$ from $x = a$ and $x = b$ and for the regions stated.

To further support the conjecture:

$$A : B$$

$$\int_{a^n}^{b^n} y^{\frac{1}{n}} : \int_a^b x^n$$

$$\left[\frac{y^{\frac{n+1}{n}}}{\frac{n+1}{n}}\right]_{a^n}^{b^n} : \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$\left(\frac{b^{n+1}}{\frac{n+1}{n}}\right)-\left(\frac{a^{n+1}}{\frac{n+1}{n}}\right):\left(\frac{b^{n+1}}{n+1}\right)-\left(\frac{a^{n+1}}{n+1}\right)$$

$$n(b^{n+1}-a^{n+1}): b^{n+1}-a^{n+1}$$

$$n:1$$