

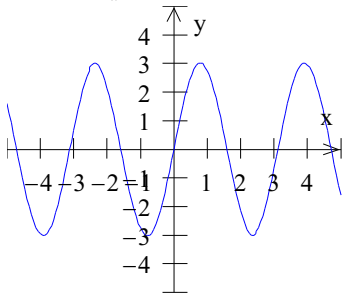
In this assignment we are being asked to discover a way to evaluate definite integrals independently of our calculator. In order to proceed with the investigation, we must identify the meaning of a definite integral.

A definite integral is defined as the area between a curve in the given x-axis in a given interval. The definite integral of the function f from $x = a$ to $x = b$ gives a way to find the product of $(b-a)$ and $f(x)$, even if f is not a constant. The definite integral is also defined as an average rate of change and can be written as:

$$\int_a^b f(x) dx$$

Given the information above, investigate the definite integral of $3\sin(2x)$ defined as:

$$I(b) = \int_a^b 3\sin(2x) dx$$



← Graph of $3\sin(2x)$

We can begin the investigation by plotting points on a graph to represent values of the definite integral. For this trial we will keep $A=0$ and B will be manipulated.

Start by plugging in the equation to the calculator:

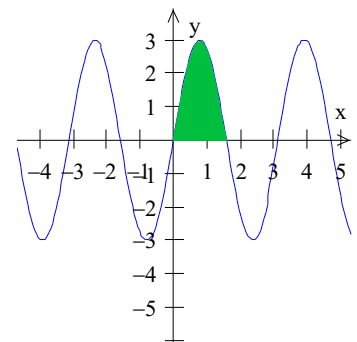
Ex. $3\sin(2x)$ where $x = \pi/2$

$$3\sin(2 \times \pi/2) = 0$$

Ex. $3\sin(2x)$ where $x = \pi/6$

$$3\sin(2 \times \pi/6) = 2.60$$

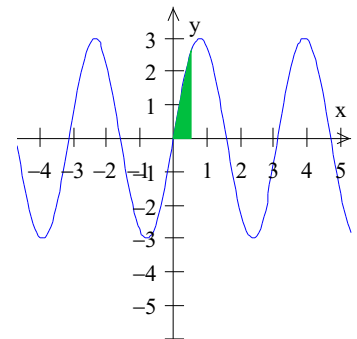
$$\int_0^{\pi/2} 3\sin(2x) dx = 3$$



These are the y values for the function. Then graph the (x,y) coordinates. The area above the x -axis and in between the two points is considered the definite integral or area under the curve. Next, shade the area under the curve from 0 to $\pi/2$ which represents:

$$\int_0^{\pi/2} 3\sin(2x) dx$$

$$\int_0^{\pi/6} 3\sin(2x) dx = 0.75$$



There is another example of a shaded region for the b value $\pi/6$. Through this investigation we are trying to estimate these areas under the curve for given values to help lead us to a formula that will work for all values and all functions.

Instead of counting squares, using the trapezoidal method, or the midpoint method, we can estimate the value of the area/definite integral on the calculator.

To get the value of the area (definite integral)

Plug **FnInt (Y₁, X, A, B)** into the graphing calculator:

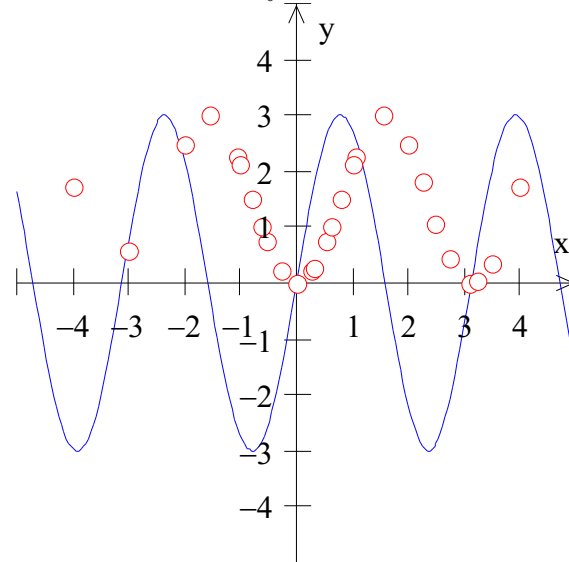
Ex. FnInt (3sin(2x), x, 0, $\pi/2$)= 3

After determining one of the definite integrals for a value of b, we must continue this process to find several points for different values of b to try to find a pattern. (Only some points are shown in the table, more points are included on the graph to get a better understanding).

B (A=0)	Y
0	0
$\pi/2$	3
$\pi/3$	2.25
$\pi/6$	0.75
$\pi/12$	0.201
$-\pi/2$	3
$-\pi/3$	2.25
$-\pi/6$	0.75
$-\pi/12$	0.201
1	2.12
2	2.48
3	0.597
4	1.72
-1	2.12
-2	2.48
-3	0.597
-4	1.72
2.25	1.82
2.5	1.08
2.75	0.437
3.25	0.035
3.5	0.369

Here we are comparing the graph of 3sin(2x) to

the graph of $I(b) = \int_0^b 3 \sin(2x) dx$



All the points are now combined to create a single graph. We can start looking for patterns by observing the values for the graph of $I(b)$ over the interval

$0 < b < 5$ (approximate values)

$0 < b < 1.6$ & $3.2 < b < 4.8$ the graph is increasing

$1.6 < b < 3.2$ & $4.8 < b < 5$ the graph is decreasing

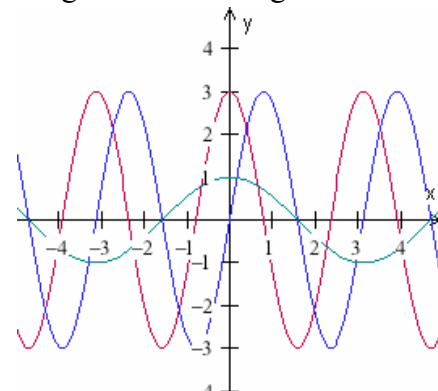
$0.8 < b < 1.6$ & $2.4 < b < 3.2$ & $4 < b < 4.8$ the graph is increasing at a decreasing rate

$0 < b < 0.8$ & $1.6 < b < 2.4$ & $3.2 < b < 4$ the graph is increasing at an increasing rate

$0 < b < 0.8$ & $2.4 < b < 4.8$ the graph is concave up

$0.8 < b < 2.4$ & $4.8 < b < 5$ the graph is concave down

Using this information, we can develop a formula without an integral for $I(b)$. Trials:



$f(x) = 3\cos(2x)$ (red)
 $f(x) = \cos(x)$ (green)
 $f(x) = 3\sin(2x)$ (blue) (original function)

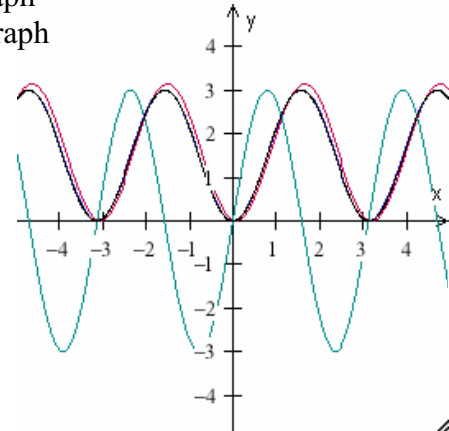
$f(x) = \pi/2\cos(2x+3) + \pi/2$ (red)
 $f(x) = 3\cos(x+\pi/2)^2$ (dark blue)
 $f(x) = 3\sin(x)^2$ (dark blue)
 $f(x) = -3/2\cos(2x)+1.5$ (dark blue)
 $f(x) = 3\sin(2x)$ (green) (original)

* Compare the values from the three dark blue functions to those of the original function. They appear to be *equivalent*. (Answers are rounded to 3 significant figures)

B (A=0)	Y
-3	0.060
-2.8	0.337
-2.6	0.797
-2.4	1.37
-2.2	1.96
-2	2.48
-1.8	2.85
-1.6	2.10
-1.4	2.91
-1.2	2.61
-1	2.12
-0.8	1.54
-0.6	0.956
-0.4	0.455
-0.2	0.118
0	0
0.2	0.118
0.4	0.455
0.6	0.960
0.8	1.54
1	2.12
1.2	2.61
1.4	2.91
1.6	2.10
1.8	2.85
2	2.48
2.2	1.96
2.4	1.37
2.6	0.797
2.8	0.337
3	0.060

Graphs of integral-less functions

1st 3 = 1st graph
 2nd 4 = 2nd graph



The equation $f(x) = -3/2\cos(2x)+1.5$ is the best function because it anti-differentiates the function so your graph becomes equivalent to the definite integral for the function $3\sin(2x)$.

$\int_a^b 3\sin(2x) dx$ is a definite integral

$\int_a^b 3\sin(2x) dx$ is equal to the $\int 3\sin(2x)$

\int is the symbol for the anti-derivative. The antiderivative of a function $f(x)$ is another function $F(x)$, whose derivative $F'(x) = f(x)$

Generalized form of the antiderivative of $f(x)$ with respect to x :

$f(x) = F(x) + C$ where C is a constant

$\int_a^b 3\sin(2x) dx = -3/2\cos(2x) + C$

Therefore: $\int_a^b 3\sin(2x) dx = -3/2\cos(2x) + 1.5$ because that is the anti-derivative of the definite integral.

The conjecture holds for all values of b for this equation. If the equation is truly a solution then it MUST hold for ALL values of b . The other computed equations do hold many of the values of b and appear to be equivalent but they are not the anti-derivative of $I(b)$

Now conduct a similar investigation using the same definite integral

$$I(b) = \int_a^b 3 \sin(2x) dx \text{ but now manipulate the A values.}$$

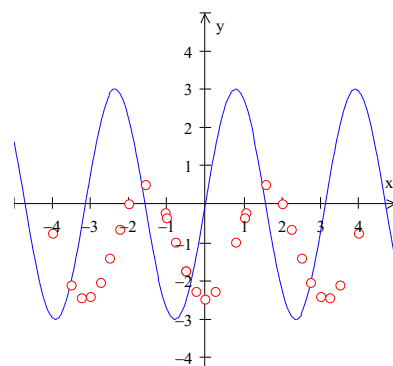
In this case we will keep the value of A constant and change the values of b and repeat this several times for different values of A.

To get the values to plot use the graphing calculator:

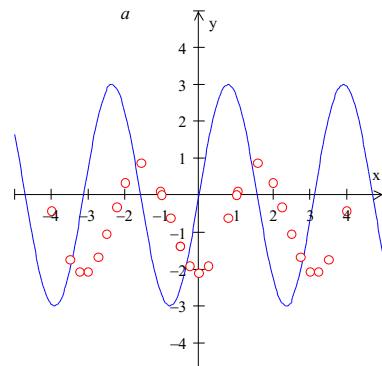
FnInt (Y₁, X, A, B)

Ex. FnInt (3sin(2x), x, -1, B)

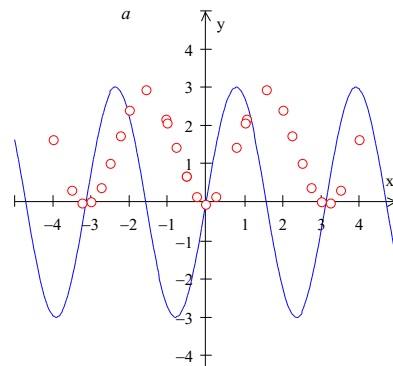
B	Y (A=± 1)	Y (A=± 2)	Y (A=± 3)	Y (A=± π/2)
0	-2.12	-2.48	-0.06	-3
π/2	0.876	0.520	2.94	0
π/3	0.126	-0.23	2.19	-0.75
π/4	-0.624	-0.98	1.44	-1.5
π/6	-1.37	-1.73	0.69	-2.25
π/12	-1.92	-2.28	0.141	-2.80
-π/2	0.876	0.520	2.94	0
-π/3	0.126	-0.230	2.19	-0.75
-π/4	-0.624	-0.980	1.44	-1.5
-π/6	-1.37	-1.73	0.69	-2.25
-π/12	-1.92	-2.28	0.141	-2.80
1	0	-0.356	2.06	-0.876
2	0.356	0	2.421	-0.520
3	-2.06	-2.42	0	-2.94
4	-0.406	-0.762	1.66	-1.28
-1	0	-0.356	2.06	-0.876
-2	0.356	0	2.42	-0.520
-3	-2.06	-2.42	0	-2.94
-4	-0.406	-0.762	1.66	-1.28
2.25	-0.308	-0.664	1.76	-1.18
2.5	-1.05	-1.41	1.01	-1.93
2.75	-1.69	-2.04	0.377	-2.56
3.25	-2.09	-2.45	-0.025	-2.96
3.5	-1.76	-2.11	0.309	-2.63
-2.25	-0.308	-0.664	1.76	-1.18
-2.5	-1.05	-1.41	1.01	-1.93
-2.75	-1.69	-2.04	0.377	-2.56
-3.25	-2.09	-2.45	-0.024	-2.96
-3.5	-1.76	-2.11	0.309	-2.63



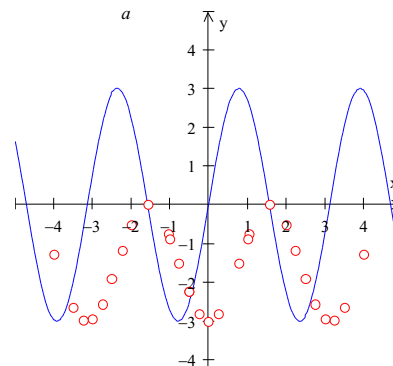
Graph of $\int_a^b 3 \sin(2x) dx$ for $a = \pm 1$



Graph of $\int_a^b 3 \sin(2x) dx$ for $a = \pm 2$



Graph of $\int_a^b 3 \sin(2x) dx$ for $a = \pm 3$



Graph of $\int_a^b 3 \sin(2x) dx$ for $a = \pm \pi/2$

Now we must try to generate a formula for $\int_a^b 3 \sin(2x) dx$ in terms of A and B. In the first steps we only made a formula in terms of b. Now we are taking into account both variables to try and reach a more general solution.

Trials:

Start by trying to substitute x for A and B.

Ex. Where A=1 and B= $\pi/3$

-3/2cos(2x)+1.5 is our general formula right now

$$-3/2\cos(2 \times \pi/3) + 1.5 = 2.25$$

$$-3/2\cos(2 \times 1) + 1.5 = 2.12$$

$$\int_1^{\pi/3} 3 \sin(2x) dx = 0.126 \quad (\text{We are given this information from the table above})$$

Ex. Where A=1 and B= $\pi/2$

$$-3/2\cos(2 \times \pi/2) + 1.5 = 3$$

$$-3/2\cos(2 \times 1) + 1.5 = 2.12$$

$$\int_1^{\pi/2} 3 \sin(2x) dx = 0.876 \quad (\text{We are given this information from the table above})$$

From these values we can try subtracting A-B to see if the value will be the same as the value from the definite integral I(b)

$$2.12 - 2.25 = -1.3$$

$$2.12 - 3 = -0.88$$

These values are similar to the actual value but are the opposite sign. If we subtract B-A instead of A-B, we will get the correct value.

$$2.25 - 2.12 = 1.3 \approx 1.26$$

$$3 - 2.12 = 0.88 \approx 0.876$$

If we know that by subtracting the values of the two functions from one another then the formula for $\int_a^b 3 \sin(2x) dx$ in terms of a and b is **$3\cos(B + \pi/2)^2 - 3\cos(A + \pi/2)^2$**

Further Examples:

$$\int_2^{\pi/3} 3 \sin(2x) dx = 3\cos(\pi/3 + \pi/2)^2 - 3\cos(2 + \pi/2)^2 = -.230$$

$$\int_3^{\pi/6} 3 \sin(2x) dx = 3\cos(\pi/6 + \pi/2)^2 - 3\cos(3 + \pi/2)^2 = .690$$

$$\int_2^{\pi/3} 3 \sin(2x) dx = 3\cos(\pi/12 + \pi/2)^2 - 3\cos(\pi/2 + \pi/2)^2 = -2.80$$

In order to reach a more generic function we need to investigate more than one function.

For a second function we will use $\int_a^b (5 - 4x)$

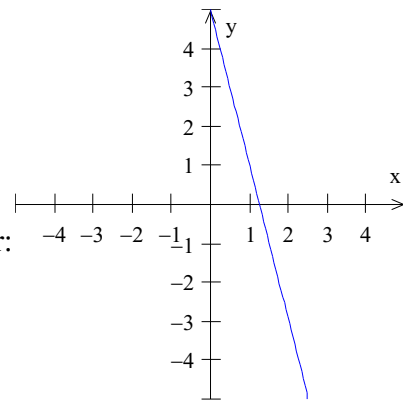
Start by graphing the function without the integral.

To get the value of the area (definite integral)

Plug **FnInt (Y₁, X, A, B)** into the graphing calculator:

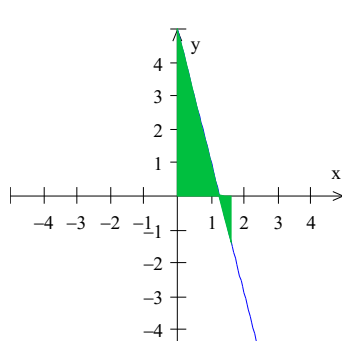
Ex. FnInt (5-4x, x, 0, $\pi/2$)= 2.92

Ex. FnInt (5-4x, x, 0, $\pi/6$)= 2.07

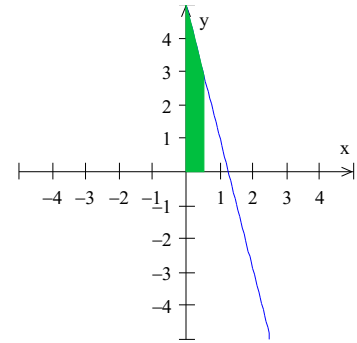


Graph of 5-4x)

B	Y (A=0)
0	0
$\pi/2$	2.92
$\pi/3$	3.04
$\pi/4$	2.69
$\pi/6$	2.07
$\pi/12$	1.17
$-\pi/2$	-12.8
$-\pi/3$	-7.43
$-\pi/4$	-5.02
$-\pi/6$	-3.17
$-\pi/12$	-1.45
1	3
2	2
3	-3
4	-12
-1	-7
-2	-18
-3	-33
-4	-52
2.25	1.13
2.5	0
2.75	-1.38
3.25	-4.88
3.5	-7
-2.25	-21.4
-2.5	-25
-2.75	-28.9
-3.25	-37.4
-3.5	-42

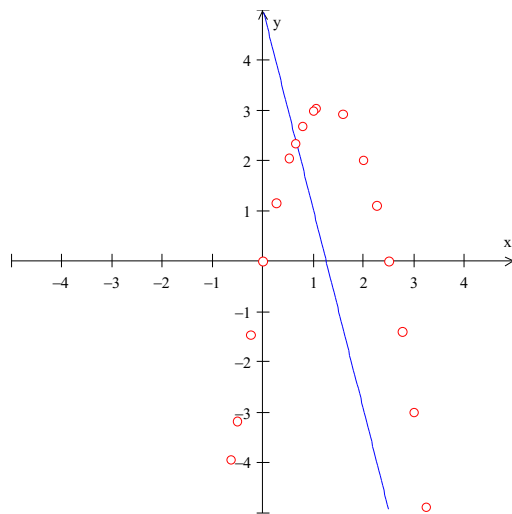


Graph of $\int_0^{\pi/2} 5 - 4x = 2.92$



Graph of $\int_0^{\pi/6} 5 - 4x = 2.07$

Both are graphs of the shaded region that represents the area under the curve or the value of the definite integral.



Graph of $\int_0^b (5 - 4x)$ for all different values of B

Observe the values for the graph of $I(b)$ over the interval $0 < b < 5$ (approximate values)

$0 < b < 1.25$ The graph is increasing

$1.25 < b < 5$ The graph is decreasing

$0 < b < 1.25$ The graph is increasing at a decreasing rate

$1.25 < b < 5$ The graph is decreasing at an increasing rate

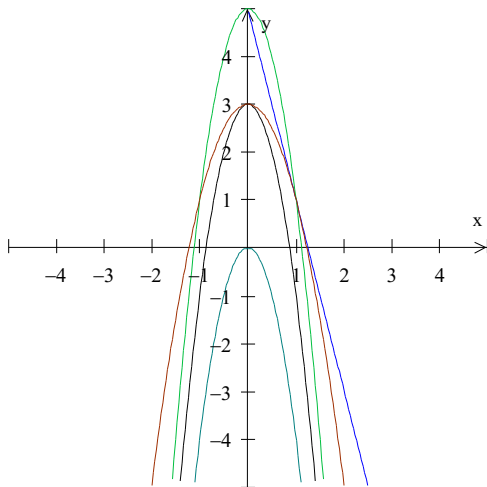
$0 < b < 5$ The graph is concave down

Now we must try to develop a formula that is equivalent to the definite integral but is integral-less.

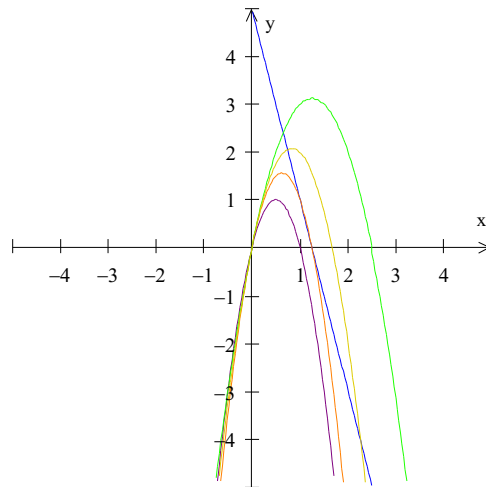
Trials:

We know the derivative must be a negative parabola because it is concave down and because it forms a parabola there has to be a squared power in the function.

$5-4x$ (blue)
 $5-4x^2$ (green)
 $-4x^2$ (turquoise)
 $-4x^2+3$ (black)
 $-4/2x^2+3$ (red)



$5-4x$ (blue)
 $5x-4x^2$ (orange)
 $4x-4x^2$ (purple)
 $5x-3x^2$ (yellow)
 $5x-2x^2$ (green)



The equation $5b-2b^2$ is the best function. This function is the anti-derivative of $5-4x$ so we know that this is the solution. The function $5b-2b^2$ holds for all values of B which also confirms that function is indeed correct.

Ex. $5(-1) - 2(-1)^2 = -7$

Ex. $5(\pi/2) - 2(\pi/2)^2 = 2.92$

Ex. $5(2.25) - 2(2.25)^2 = 1.13$

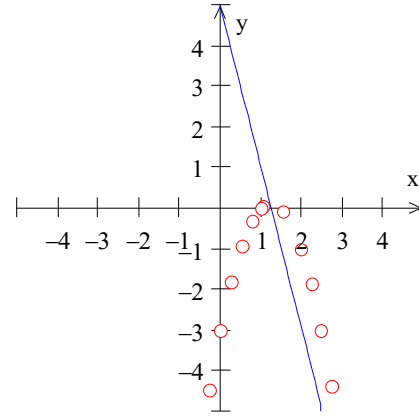
Now we can use different values of A to try and find a more general formula:

B	Y (A=1)	Y (A=-1)	Y (A= $\pi/2$)
0	-3	7	-2.92
$\pi/2$	-0.081	9.92	0
$\pi/3$	0.043	10	0.124
$\pi/4$	-0.307	9.69	-0.226
$\pi/6$	-0.93	9.07	-0.849
$\pi/12$	-1.828	8.17	-1.75
$-\pi/2$	-15.8	-5.79	-15.7
$-\pi/3$	-10.4	-0.429	-10.3
$-\pi/4$	-8.16	1.84	-8.08
$-\pi/6$	-6.17	3.83	-6.09
$-\pi/12$	-4.45	5.55	-4.37
1	0	10	0.081
2	-1	9	-0.92
3	-6	4	-5.92
4	-15	-5	-14.9
-1	-10	0	-9.92
-2	-21	-11	-20.9
-3	-36	-26	-35.9
-4	-55	-45	-54.9
2.25	-1.88	8.125	-1.79
2.5	-3	7	-2.92
2.75	-4.38	5.63	-4.29
3.25	-7.88	2.13	-7.79
3.5	-10	0	-9.92
-2.25	-24.4	-14.4	-24.29
-2.5	-28	-18	-27.92
-2.75	-31.8	-21.9	-31.79
-3.25	-40.38	-30.4	-40.29
-3.5	-45	-35	-44.92

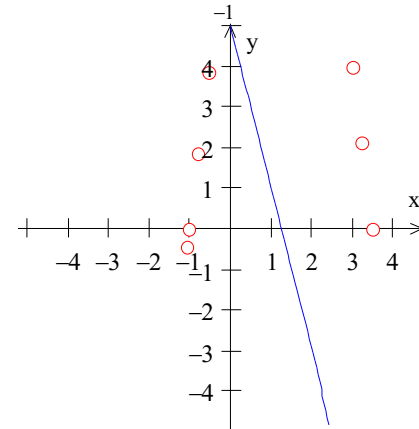
With our previous knowledge from the Previous function, we can predict what The new (more generic) function may Look like.

$$\text{Given } \int_a^b (5 - 4x) = 5x - 4x^2$$

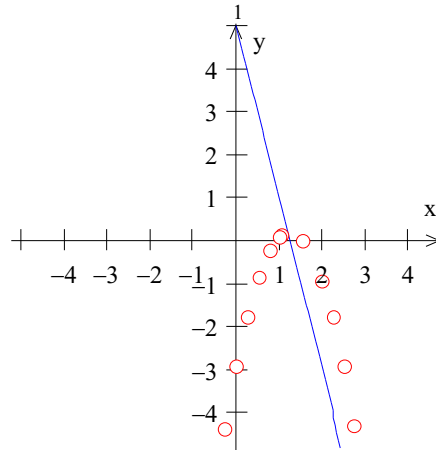
Let $(5b - 4b^2) - (5a - 4a^2)$ be the function



Graph of $\int_0^b (5 - 4x)$



Graph of $\int_{-1}^b (5 - 4x)$



Graph of $\int_{\pi/2}^b (5 - 4x)$

Plug in some examples to see if the same formula applies to this function.

$$\text{Ex. } \int_1^1 (5 - 4x) = ((5 \times 1) + 4(1^2)) - ((5 \times 1) + 4(1^2)) = 0$$

$$\text{Ex. } \int_{\pi/2}^1 (5 - 4x) = ((5 \times 1) + 4(1^2)) - ((5 \times \pi/2) + 4(\pi/2^2)) = 0.081$$

$$\text{Ex. } \int_{-1}^{\pi/2} (5 - 4x) = ((5 \times \pi/2) + 4(\pi/2^2)) - ((5 \times -1) + 4(-1^2)) = 9.92$$

From this information, we can formulate a shortcut to find $\int_a^b f(x) dx$ for any generic function $f(x)$.

$$\int_a^b f(x) dx = \int f(b) - \int f(a) \text{ Where } \int \text{ is the symbol for the anti-derivative}$$

In order to ensure that this shortcut is correct, we must test it out with other example functions. We can try other functions that would require us to apply some of the anti-differentiation rules such as the anti-power rule $= \frac{x^{n+1}}{n+1}$

$$\int_a^b f(x^2) dx \text{ The anti-derivative of this function} = 1/3 x^3$$

$$\int_0^1 f(x^2) dx = ((1/3)(1^3)) - ((1/3)(0^3)) = .333 - 0 = .333$$

$$\int_1^2 f(x^2) dx = ((1/3)(2^3)) - ((1/3)(1^3)) = 2.67 - .333 = 2.33$$

$$\begin{aligned} \int_1^2 f\left(\frac{x^3 - 2}{x^3}\right) dx &= \int (x^3 - 2)x^3 dx = \int (1 - 2x^{-3}) dx \\ &= x - \frac{2x^{-2}}{-2} + c = x + x^{-2} + c \\ &= \frac{x^3}{x^2} + \frac{1}{x^2} + \frac{c}{x^2} = \frac{x^3 + 1 + c}{x^2} \end{aligned}$$

Now substitute X for values of A and B

$$\frac{A^3 + I + C A^2}{A^2}$$

$$\frac{B^3 + I + C B^2}{B^2}$$

Apply the general formula where $\int_a^b f(x)dx = \int f(b) - \int f(a)$

$$\frac{B^3 + I + C B^2}{B^2} - \frac{A^3 + I + C A^2}{A^2}$$

A=1 and B=2

$$\frac{2^3 + I + C 2^2}{2^2} - \frac{1^3 + I + C 1^2}{1^2} = \frac{8 + I + 4C}{4} - \frac{1 + I + 4C}{1} = \frac{I}{4} = .25$$

Now plug in on the calculator:

FnInt((x³-2)/(x³), x, 1, 2

this calculation equals .25 so the generic function does work for this equation.

Assuming that the function of the definite integral itself can be anti-differentiated, then the generic formula we found $\int_a^b f(x)dx = \int f(b) - \int f(a)$ should work for all functions at

any values of A and B. This helps us to establish a universal rule that can be applied to several different functions. Through this investigation, we proved that you could find a pattern for a mathematical concept by investigating a functions properties and applying our knowledge of calculus and other mathematical concepts to reach an answer. We applied what we already knew about definite integrals, derivatives, anti-derivatives, etc. to develop a formula that now works for all generic functions when you are trying to derive the anti-derivative that is equivalent to the value of the definite integral.