Marked by Teachers

This document was downloaded from www.markedbyteachers.com

1.

Area A formed in the function $y=x^2$ from y=0 to y=1 and the y-axis can be found by integrating the function with respect to y.

Area A:

$y=x^2$

$x=\sqrt{y}$

$x=y^{\frac{1}{2}}$

$\int_0^1 y^{\frac{1}{2}}\,dy$

$=[y^{\frac{3}{2}} \div \frac{3}{2}]_0^1$

$=[\frac{2}{3}y^{\frac{3}{2}}]_0^1$

$=(\frac{2}{3} \times 1^{\frac{3}{2}}) - (\frac{2}{3} \times 0^{\frac{3}{2}})$

$=\frac{2}{3}$ units^2

Area B formed in the function $y=x^2$ from x=0 to x=1 and the x-axis can be found by integrating the function with respect to x.

Area B:

$y=x^2$

$\int_0^1 x^2\,dx$

$=[x^3 \div 3]_0^1$

$=\left[\frac{x^3}{3}\right]_0^1$

$=(\frac{1^3}{3} - \frac{0^3}{3})$

$=\frac{1}{3}$ units^2

Therefore, the ratio of the areas A and B for the function $y=x^2$ is:

A : B = $\frac{2}{3} : \frac{1}{3}$

= <u>2 : 1</u>

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

Now try the same with the function $y=x^n$ when $n \in Z^+$ and between x=0 and x=1.

Area A formed in the function from y=0 and $y=1^n$ and the y-axis can be found by integrating the function in respect for y.

Area A:

$y=x^n$

$x=y^{\frac{1}{n}}$

$\int_0^1 y^{\frac{1}{n}}\,dy$

$=[y^{\frac{1}{n}+1} \div \frac{1+n}{n}]_0^1$

$=[\frac{n}{n+1} y^{\frac{1}{n}+1}]_0^1$

$=(\frac{n}{n+1} \times 1^{\frac{1}{n}+1}) - (\frac{n}{n+1} \times 0^{\frac{1}{n}+1})$

$=\frac{n}{n+1}$ units^2

Area B formed in the function from x=0 and x=1 and the x-axis can be found by integrating the function in respect of x.

Area B:

$y=x^n$

$\int_0^1 x^n\,dx$

$=[x^{n+1} \div (n+1)]_0^1$

$=\left[\frac{x^{n+1}}{n+1}\right]_0^1$

$=(\frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1})$

$=\frac{1}{n+1}$ units^2

Therefore, the ratio of the areas A and B for the function $y=x^n$ is:

A : B = $\frac{n}{n+1} : \frac{1}{n+1}$

= <u>n : 1</u>

<u>Conjecture for the ratio of the areas A and B for the function $y=x^n$ is A : B = n : 1.</u>

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

Now, try and test the conjecture for other subsets of the real numbers.
When n=3,

Area A:

$y=x^3$

$x=y^{\frac{1}{3}}$

$\int_0^1 y^{\frac{1}{3}}\,dy$

$=[y^{\frac{1}{3}+1} \div \frac{1+3}{3}]_0^1$

$=[\frac{3}{4}y^{\frac{4}{3}}]_0^1$

$=(\frac{3}{4} \times 1^{\frac{4}{3}}) - (\frac{3}{4} \times 0^{\frac{4}{3}})$

$=\frac{3}{4}$ units^2

Area B:

$y=x^3$

$\int_0^1 x^3\,dx$

$=[x^{3+1} \div (3+1)]_0^1$

$=\left[\frac{x^4}{4}\right]_0^1$

$=(\frac{1^4}{4} - \frac{0^4}{4})$

$=\frac{1}{4}$ units^2

Therefore, the ratio of the areas A and B in the function $y=x^3$ is:

A : B = $\frac{3}{4} : \frac{1}{4}$

= <u>3 : 1</u>

In $y=x^3$, the value of n is 3, and therefore the conjecture n : 1 works when n=3.

When n=10,

Area A:

$y=x^{10}$

$x=y^{\frac{1}{10}}$

$\int_0^1 y^{\frac{1}{10}}\,dy$

$=[y^{\frac{1}{10}+1} \div \frac{1+10}{10}]_0^1$

$=[\frac{10}{11}y^{\frac{11}{10}}]_0^1$

$=(\frac{10}{11} \times 1^{\frac{11}{10}}) - (\frac{10}{11} \times 0^{\frac{11}{10}})$

$=\frac{10}{11}$ units^2

Area B:

$y=x^{10}$

$\int_0^1 x^{10}\,dx$

$=[x^{10+1} \div (10+1)]_0^1$

$=\left[\frac{x^{11}}{11}\right]_0^1$

$=(\frac{1^{11}}{11} - \frac{0^{11}}{11})$

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

$=\frac{1}{11}$ units^2

Therefore, the ratio of the areas A and B in the function $y=x^{10}$ is:

A : B = $\frac{10}{11} : \frac{1}{11}$

= <u>10 : 1</u>

In $y=x^{10}$, the value of n is 10, and therefore the conjecture n : 1 works when n=10.

When n=47,

Area A:

$y=x^{47}$

$x=y^{\frac{1}{47}}$

$\int_0^1 y^{\frac{1}{47}}\,dy$

$=[y^{\frac{1}{47}+1} \div \frac{1+47}{47}]_0^1$

$=[\frac{47}{48}y^{\frac{48}{47}}]_0^1$

$=(\frac{47}{48} \times 1^{\frac{48}{47}}) - (\frac{47}{48} \times 0^{\frac{48}{47}})$

$=\frac{47}{48}$ units^2

Area B:

$y=x^{47}$

$\int_0^1 x^{47}\,dx$

$=[x^{47+1} \div (47+1)]_0^1$

$=\left[\frac{x^{48}}{48}\right]_0^1$

$=(\frac{1^{48}}{48} - \frac{0^{48}}{48})$

$=\frac{1}{48}$ units^2

Therefore, the ratio of the areas A and B in the function $y=x^{47}$ is:

A : B = $\frac{47}{48} : \frac{1}{48}$

= <u>47 : 1</u>

In $y=x^{47}$, the value of n is 47, and therefore the conjecture n : 1 works when n=47.

When n=113,

Area A:

$y=x^{113}$

$x=y^{\frac{1}{113}}$

$\int_0^1 y^{\frac{1}{113}}\,dy$

$=[y^{\frac{1}{113}+1} \div \frac{1+113}{113}]_0^1$

$=[\frac{113}{114}y^{\frac{114}{113}}]_0^1$

$=(\frac{113}{114} \times 1^{\frac{114}{113}}) - (\frac{113}{114} \times 0^{\frac{114}{113}})$

$=\frac{113}{114}$ units^2

Area B:

$y=x^{113}$

$\int_0^1 x^{113}\,dx$

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

$=[x^{113+1} \div (113+1)]_0^1$

$=\left[\frac{x^{114}}{114}\right]_0^1$

$=(\frac{1^{114}}{114} - \frac{0^{114}}{114})$

$=\frac{1}{114}$ units^2

Therefore, the ratio of the areas ⮝ and B in the function $y=x^{113}$ is:

⮝ : B = $\frac{113}{114} : \frac{1}{114}$

= <u>113 : 1</u>

In $y=x^{113}$, the value of n is 113, and therefore the conjecture n : 1 works when n=113.

<u>∴ It can be assumed that the conjecture n : 1 works in any positive real numbers.</u>

2. To determine whether the conjecture only holds for areas between x=0 and x=1, now try the same thing but changing the values of x.
   When x is from 0 to 2,

   ⮝rea ⮝:

   $y=x^n$

   $x=y^{\frac{1}{n}}$

   $\int_0^{2^n} y^{\frac{1}{n}}\, dy$

   $=[y^{\frac{1}{n}+1} \div \frac{1+n}{n}]_0^{2^n}$

   $=[\frac{n}{n+1} y^{\frac{n+1}{n}}]_0^{2^n}$

   $=(\frac{n}{n+1} \times 2^{n\frac{n+1}{n}}) - (\frac{n}{n+1} \times 0^{\frac{n+1}{n}})$

   $=2^{n+1}\frac{n}{n+1}$ units^2

   ⮝rea B:

   $y=x^n$

   $\int_0^2 x^n\, dx$

   $=[x^{n+1} \div (n+1)]_0^2$

   $=\left[\frac{x^{n+1}}{n+1}\right]_0^2$

   $=(\frac{2^{n+1}}{n+1} - \frac{0^{n+1}}{n+1})$

   $=\frac{2^{n+1}}{n+1}$ units^2

   Therefore, the ratio of the areas ⮝ and B when x is between 0 and 2 is:

   ⮝ : B = $2^{n+1}\frac{n}{n+1} : \frac{2^{n+1}}{n+1}$

   = <u>n : 1</u>..... Conjecture works

   When x is from 1 to 2,

   ⮝rea ⮝:

   $y=x^n$

   $x=y^{\frac{1}{n}}$

   $\int_{1^n}^{2^n} y^{\frac{1}{n}}\, dy$

   $=[y^{\frac{1}{n}+1} \div \frac{1+n}{n}]_{1^n}^{2^n}$

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

$=[\frac{n}{n+1}y^{\frac{n+1}{n}}]_{1^n}^{2^n}$

$=(\frac{n}{n+1}\times 2^{n\frac{n+1}{n}})-(\frac{n}{n+1}\times 1^{n\frac{n+1}{n}})$

$=2^{n+1}\frac{n}{n+1}-\frac{n}{n+1}$

$=\frac{n}{n+1}(2^{n+1}-1)$ units^2

$\int_1^2 x^n\,dx$

$=[x^{n+1}\div(n+1)]_1^2$

$=\left[\frac{x^{n+1}}{n+1}\right]_1^2$

$=(\frac{2^{n+1}}{n+1}-\frac{1^{n+1}}{n+1})$

$=\frac{2^{n+1}}{n+1}-\frac{1}{n+1}$

$=\frac{1}{n+1}(2^{n+1}-1)$ units^2

Area B:

$y=x^n$

Therefore, the ratio of the areas A and B when x is between 0 and 2 is:

A : B = $\frac{n}{n+1}(2^{n+1}-1)\ :\frac{1}{n+1}(2^{n+1}-1)$

= <u>n : 1</u>..... Conjecture works

When x is from -10 to 2,

Area A:

$y=x^n$

$x=y^{\frac{1}{n}}$

$\int_{(-10)^n}^{2^n} y^{\frac{1}{n}}\,dy$

$=[y^{\frac{1}{n}+1}\div\frac{1+n}{n}]_{(-10)^n}^{2^n}$

$=[\frac{n}{n+1}y^{\frac{n+1}{n}}]_{(-10)^n}^{2^n}$

$=(\frac{n}{n+1}\times 2^{n\frac{n+1}{n}})-(\frac{n}{n+1}\times(-10)^{n\frac{n+1}{n}})$

$=2^{n+1}\frac{n}{n+1}-(-10)^{n+1}\frac{n}{n+1}$

$=\frac{n}{n+1}(2^{n+1}+10^{n+1})$ units^2

Area B:

$y=x^n$

$\int_{-10}^{2} x^n\,dx$

$=[x^{n+1}\div(n+1)]_{-10}^{2}$

$=\left[\frac{x^{n+1}}{n+1}\right]_{-10}^{2}$

$=(\frac{2^{n+1}}{n+1}-\frac{(-10)^{n+1}}{n+1})$

$=\frac{1}{n+1}(2^{n+1}-(-10)^{n+1})$

$=\frac{1}{n+1}(2^{n+1}+10^{n+1})$ units^2

Therefore, the ratio of the areas A and B when x is between 0 and 2 is:

A : B = $\frac{n}{n+1}(2^{n+1}+10^{n+1})\ :\frac{1}{n+1}(2^{n+1}+10^{n+1})$

= <u>n : 1</u>..... Conjecture works

When x is from -15 to -2,

Area A:

$y=x^n$

$x=y^{\frac{1}{n}}$

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

$\int_{(-15)^n}^{(-2)^n} y^{\frac{1}{n}}\,dy$

$=[y^{\frac{1}{n}+1} \div \frac{1+n}{n}]_{(-15)^n}^{(-2)^n}$

$=[\frac{n}{n+1} y^{\frac{n+1}{n}}]_{(-15)^n}^{(-2)^n}$

$=(\frac{n}{n+1} \times (-2)^{n\frac{n+1}{n}}) - (\frac{n}{n+1} \times (-15)^{n\frac{n+1}{n}})$

$=(-2)^{n+1}\frac{n}{n+1} - (-15)^{n+1}\frac{n}{n+1}$

$=-\frac{n}{n+1}(2^{n+1} - 15^{n+1})$ units^2

$y=x^n$

$\int_{-15}^{-2} x^n\,dx$

$=[x^{n+1} \div (n+1)]_{-15}^{-2}$

$=\left[\frac{x^{n+1}}{n+1}\right]_{-15}^{-2}$

$=(\frac{(-2)^{n+1}}{n+1} - \frac{(-15)^{n+1}}{n+1})$

$=\frac{1}{n+1}((-2)^{n+1} - (-15)^{n+1})$

$=-\frac{1}{n+1}(2^{n+1} - 15^{n+1})$ units^2

Area B:

Therefore, the ratio of the areas A and B when x is between 0 and 2 is:

A : B $=-\frac{n}{n+1}(2^{n+1} - 15^{n+1})\ : -\frac{1}{n+1}(2^{n+1} - 15^{n+1})$

= <u>n : 1</u> ..... Conjecture works

When x is from 10 to 21,

Area A:

$y=x^n$

$x=y^{\frac{1}{n}}$

$\int_{10^n}^{21^n} y^{\frac{1}{n}}\,dy$

$=[y^{\frac{1}{n}+1} \div \frac{1+n}{n}]_{10^n}^{21^n}$

$=[\frac{n}{n+1} y^{\frac{n+1}{n}}]_{10^n}^{21^n}$

$=(\frac{n}{n+1} \times 21^{n\frac{n+1}{n}}) - (\frac{n}{n+1} \times 10^{n\frac{n+1}{n}})$

$=21^{n+1}\frac{n}{n+1} - 10^{n+1}\frac{n}{n+1}$

$=\frac{n}{n+1}(21^{n+1} - 10^{n+1})$ units^2

Area B:

$y=x^n$

$\int_{10}^{21} x^n\,dx$

$=[x^{n+1} \div (n+1)]_{10}^{21}$

$=\left[\frac{x^{n+1}}{n+1}\right]_{10}^{21}$

$=(\frac{21^{n+1}}{n+1} - \frac{10^{n+1}}{n+1})$

$=\frac{1}{n+1}(21^{n+1} - 10^{n+1})$ units^2

Therefore, the ratio of the areas A and B when x is between 0 and 2 is:

A : B $=\frac{n}{n+1}(21^{n+1} - 10^{n+1})\ : \frac{1}{n+1}(21^{n+1} - 10^{n+1})$

= <u>n : 1</u> ..... Conjecture works

When x is from 91 to 214,

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

Area A:

$y=x^n$

$x=y^{\frac{1}{n}}$

$\int_{91^n}^{214^n} y^{\frac{1}{n}}\,dy$

$=[y^{\frac{1}{n}+1} \div \frac{1+n}{n}]_{91^n}^{214^n}$

$=[\frac{n}{n+1} y^{\frac{n+1}{n}}]_{91^n}^{214^n}$

$=($

$\frac{n}{n+1} \times 214^{n\frac{n+1}{n}}) - (\frac{n}{n+1} \times 91^{n\frac{n+1}{n}})$

$=214^{n+1}\frac{n}{n+1} - 91^{n+1}\frac{n}{n+1}$

$=\frac{n}{n+1}(214^{n+1} - 91^{n+1})$ units^2

Area B:

$y=x^n$

$\int_{91}^{214} x^n\,dx$

$=[x^{n+1} \div (n+1)]_{91}^{214}$

$=\left[\frac{x^{n+1}}{n+1}\right]_{91}^{214}$

$=(\frac{214^{n+1}}{n+1} - \frac{91^{n+1}}{n+1})$

$=\frac{1}{n+1}(214^{n+1} - 91^{n+1})$ units^2

Therefore, the ratio of the areas A and B when x is between 0 and 2 is:

A : B = $\frac{n}{n+1}(214^{n+1} - 91^{n+1})$: $\frac{1}{n+1}(214^{n+1} - 91^{n+1})$

= <u>n : 1</u> ..... Conjecture works

<u>∴ It can be said that the conjecture works for areas between any real numbers.</u>

3. When $y=x^n$, x is between a and b and y is between a^n and b^n,

Area A:

$y=x^n$

$x=y^{\frac{1}{n}}$

$\int_{a^n}^{b^n} y^{\frac{1}{n}}\,dy$

$=[y^{\frac{1}{n}+1} \div \frac{n+1}{n}]_{a^n}^{b^n}$

$=[\frac{n}{n+1} y^{\frac{n+1}{n}}]_{a^n}^{b^n}$

$=(\frac{n}{n+1}(b^n)^{\frac{n+1}{n}} - \frac{n}{n+1}(a^n)^{\frac{n+1}{n}})$

$=\frac{b^{n+1}}{n+1}n - \frac{a^{n+1}}{n+1}n$

$=n(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1})$ units^2

Area B:

$y=x^n$

$\int_a^b x^n\,dx$

$=[\frac{x^{n+1}}{n+1}]_a^b$

$=(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1})$ units^2

Therefore, the ratio of the areas A and B is,

A : B = $n(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1})$: $(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1})$

= <u>n : 1 ..... Conjecture works</u>

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

∴The conjecture n : 1 is true for the general case $y=x^n$ from x=a to x=b where a<b and the area ⮝ is defined as $y=x^n$, $y=a^n$, $y=b^n$ and the y-axis and the area B is defined as $y=x^n$, x=a, x=b and the x-axis.

4. (a) We need to find the volume B in order to find out the volume ⮝ so we start off with volume B.

Volume B:

$y=x^n$

$\pi\int_a^b x^{2n}\,dx$

$=\pi[\frac{x^{2n+1}}{2n+1}]_a^b$

$=\pi(\frac{b^{2n+1}}{2n+1}-\frac{a^{2n+1}}{2n+1})$

$=\pi(\frac{b^{2n+1}-a^{2n+1}}{2n+1})$ units^3

⮝nd then volume ⮝.

Volume ⮝:

$b^n \times b^n \times \pi \times b - a^n \times a^n \times \pi \times a - \text{volume B}$

$=\pi b^{2n+1} - \pi a^{2n+1} - \pi(\frac{b^{2n+1}-a^{2n+1}}{2n+1})$

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

$=\frac{(2n+1)\pi b^{2n+1}-(2n+1)\pi a^{2n+1}-\pi(b^{2n+1}-a^{2n+1})}{2n+1}$

$=\frac{\pi((2n+1)b^{2n+1}-(2n+1)a^{2n+1}-(b^{2n+1}-a^{2n+1}))}{2n+1}$

$=\frac{\pi(2nb^{2n+1}+b^{2n+1}-2na^{2n+1}-a^{2n+1}-b^{2n+1}+a^{2n+1})}{2n+1}$

$=\frac{\pi(2nb^{2n+1}-2na^{2n+1})}{2n+1}$

$=\frac{2n\pi(b^{2n+1}-a^{2n+1})}{2n+1}$ units^3

Therefore, the ratio of the volumes ▲ and B is:

▲ : B = $\frac{2n\pi(b^{2n+1}-a^{2n+1})}{2n+1} : \pi(\frac{b^{2n+1}-a^{2n+1}}{2n+1})$

=2n : 1

∴ General formula for the ratios of the volumes is 2n : 1

(b)Volume ▲:

$y=x^n$

$x=y^{\frac{1}{n}}$

$\pi\int_{a^n}^{b^n} y^{\frac{2}{n}}\,dy$

$=\pi[y^{\frac{2+n}{n}} \div \frac{2+n}{n}]_{a^n}^{b^n}$

$=\pi[\frac{n}{2+n}y^{\frac{2+n}{n}}]_{a^n}^{b^n}$

$=\pi\left(\frac{n}{2+n}b^{2+n}-\frac{n}{2+n}a^{2+n}\right)$

$=\frac{\pi n}{2+n}(b^{2+n}-a^{2+n})$ units^3

Volume B:

Marked by Teachers

This document was downloaded from www.markedbyteachers.com

$b \times b \times \pi \times b^n - a \times a \times \pi \times a^n - \text{volume } A$

$= \pi b^{2+n} - \pi a^{2+n} - \frac{\pi n}{2+n}(b^{2+n} - a^{2+n})$

$= \frac{\pi b^{2+n}(2+n) - \pi a^{2+n}(2+n) - \pi n(b^{2+n} - a^{2+n})}{2+n}$

$= \frac{\pi(b^{2+n}(2+n) - a^{2+n}(2+n) - n(b^{2+n} - a^{2+n})}{2+n}$

$= \frac{\pi(2b^{2+n} + nb^{2+n} - 2a^{2+n} - na^{2+n} - nb^{2+n} + na^{2+n})}{2+n}$

$= \frac{\pi(2b^{2+n} - 2a^{2+n})}{2+n}$

$= \frac{2\pi(b^{2+n} - a^{2+n})}{2+n}$ units^3

Therefore, the ratio of the volumes A and B is:

A : B = $\frac{\pi n}{2+n}(b^{2+n} - a^{2+n}) : \frac{2\pi(b^{2+n} - a^{2+n})}{2+n}$

= <u>n : 2</u>

<u>∴ General formula for the ratios of the volumes is n : 2</u>