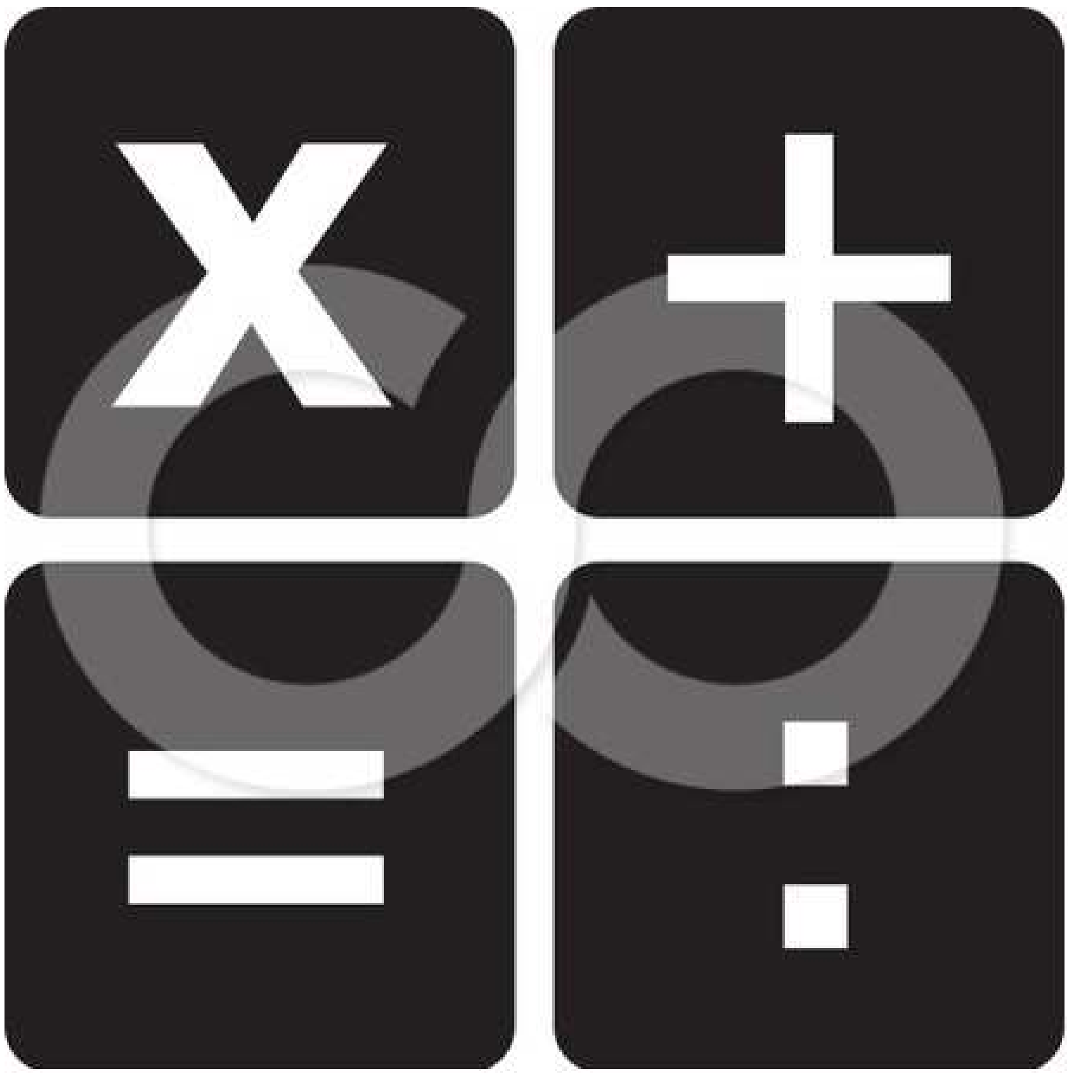
Math HL Portfolio

Parabola Investigation



INTRODUCTION

In this portfolio I am going to investigate the patterns formed when a parabola,

$y=ax^2+bx+c$ intersects with the lines $y=x$ and $y=2x$. Further I would be broadening the scope of this investigation to other lines and other orders of polynomials and observing patterns in their respective intersections.

The first part of my portfolio would include parabolas with their turning points located in the 1st quadrant of a graph. I would be investigating the patterns formed when the two lines $y=x$ and $y=2x$ intersect with parabolas that have different coefficients of x^2 but lie in the first quadrant and form a conjecture. Then I would prove it and test its limitations.

The second part would include testing and modifying my conjecture for the turning point of the parabola situated in any quadrant and testing for all real co-efficient of x^2.

The third part would include modifications to the conjecture if the intersecting lines are changed and hence finding its limitations.

In the final part of my portfolio I will also try and make a conjecture for polynomials of higher order and derive any patterns or observations plausible.

PARABOLAS IN THE FIRST QUADRANT INTERSECTING WITH y=x AND y=2x.

A polynomial is a mathematical expression involving a sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable (i.e., a univariate polynomial) with constant coefficients is given by:

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

A parabola is the name given to the shape of the graph formed with any polynomial of degree 2. This polynomial can be expressed in three forms:

- $f(x) = ax^2+bx+c$, $a\neq 0$ → This is the general form.
- $f(x)=a(x-h)^2 + k$, $a\neq 0$ → This is the vertex form where the point (h,k) is its turning point.
- $f(x)=a(x-m)(x-n)$, $a\neq 0$ → This is the factored form where m and n are the two roots of the equation.

In the 1st part of my portfolio I will only consider a>o, giving me a minimum point for each parabola rather than a maximum. The 2nd would include values of a < 0.

We know that for a graph to have its turning point in the first quadrant and a>0, it should have no real roots. This means that the parabola would not cut with the x-axis and its determinant would be greater than zero. Hence: $b^2 - 4ac > 0$,where a >0 .

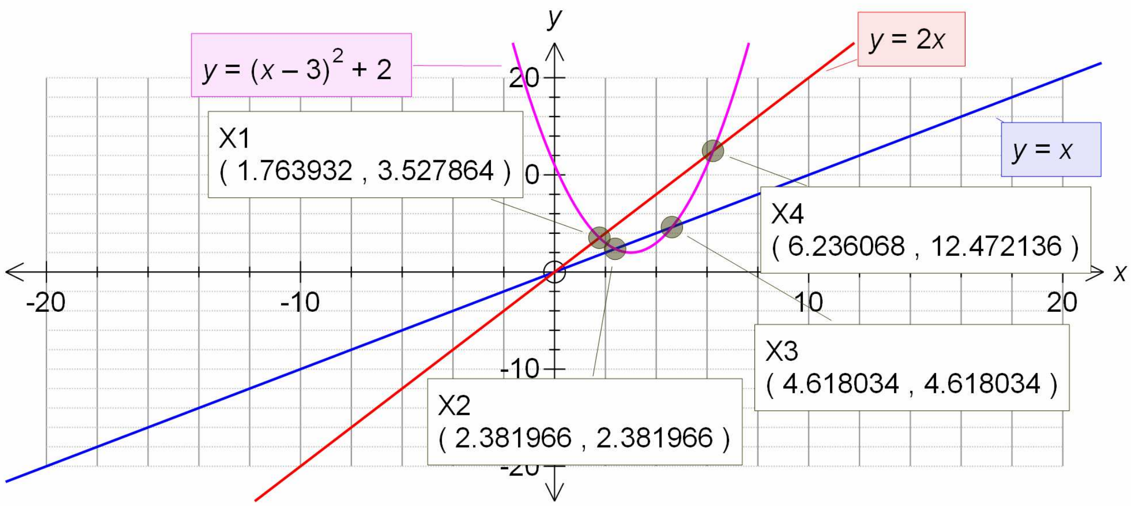
We also know that for the turning point (h,k) to be in the first quadrant both h and k must be greater than zero .

We want to find four intersection points. This means that the parabola MUST intersect with y=x. If a parabola cuts y=x, then it also cuts y=2x. For a parabola to cut y=x at two points, its turning point should be below the line. For the turning point to be below the line the x-coordinate of the vertex should be greater than the y-coordinate. This implies h>k. I am going to keep the values of h at 3 and k at 2 constant in this part, and change the values of 'a' to see any emerging pattern.

Thus our equation is :

$y = a(x-3)^2 + 2$

1. Now let us consider the parabola $y= (x-3)^2+2 = x^2-6x+11$ and the lines $y=x$ and $y=2x$.
   The vertex of this equation is (3, 2) and the value, a = 1. Thus, this parabola has its turning point in the first quadrant.
2. Using the software '*Fx Graph4*' I have plotted these 3 functions on a graph.
3. Then I have found the four intersection points of these three functions and labeled them as X_1, X_2, X_3, X_4 respectively from left to right. (see Graph 1)



Graph 1: Value of a=1

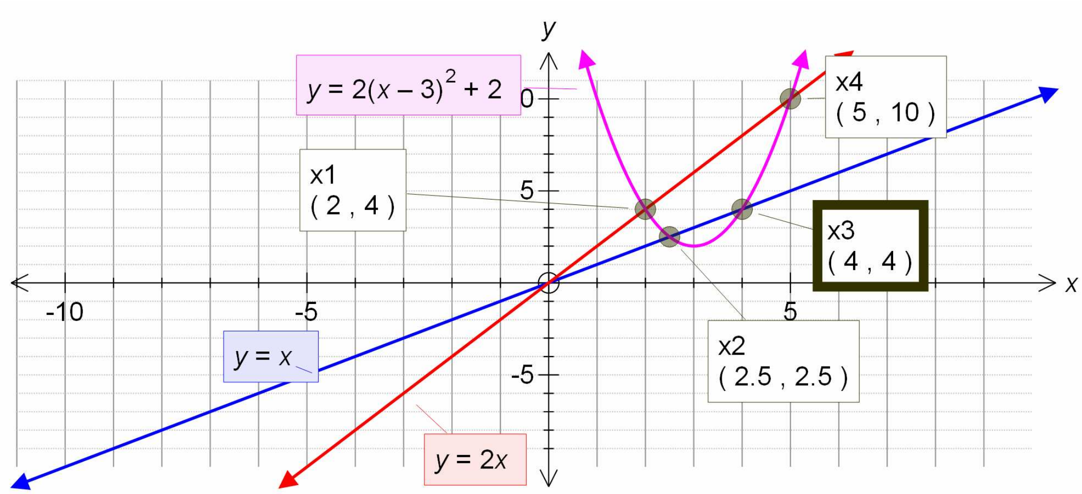
4. Then I found out the values of $(X_2 – X_1)$ and $(X_4 – X_3)$ and called them S_L and S_R respectively.
5. I finally calculated the value of $D = |S_L – S_R|$

These results are illustrated in table 1.

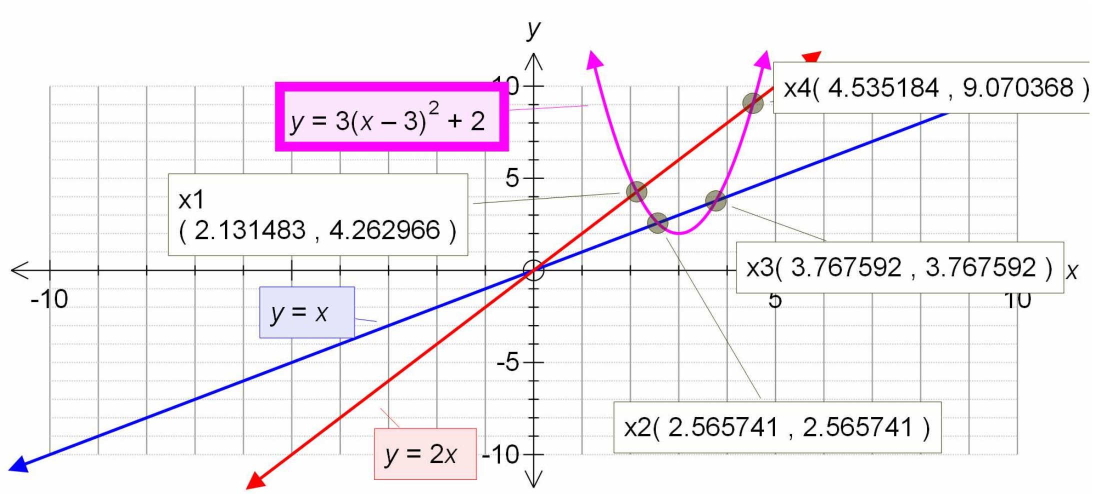
A	X_1	X_2	X_3	X_4	$S_L = X_2 - X_1$	$S_R = X_4 - X_3$	$D=\|S_L-S_R\|$
1	1.763932	2.381966	4.618034	6.236068	0.618034	1.618034	1

(software used: "Microsoft Excel")

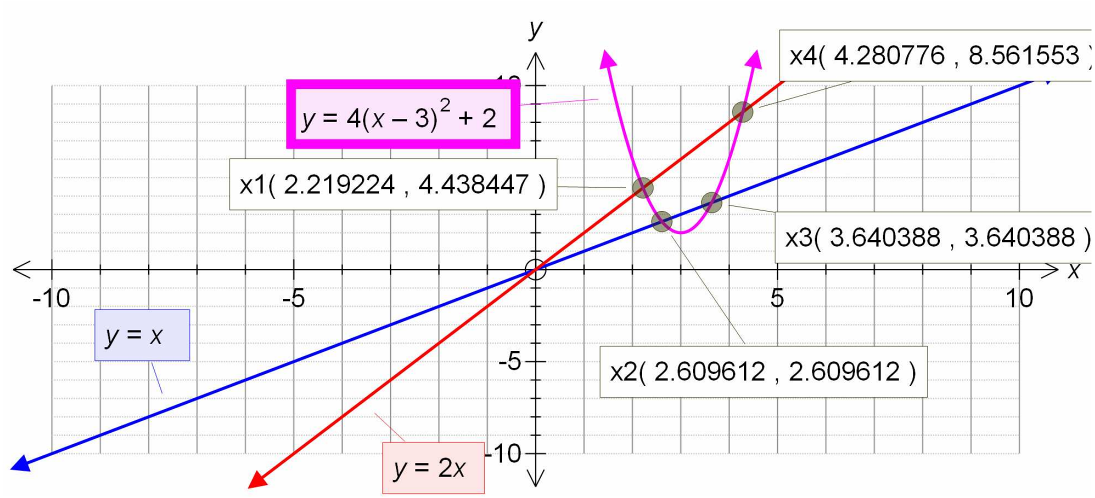
I am now going to repeat steps 1 to 5 for different values of a, all being greater than 0 and all belonging to Z^+.



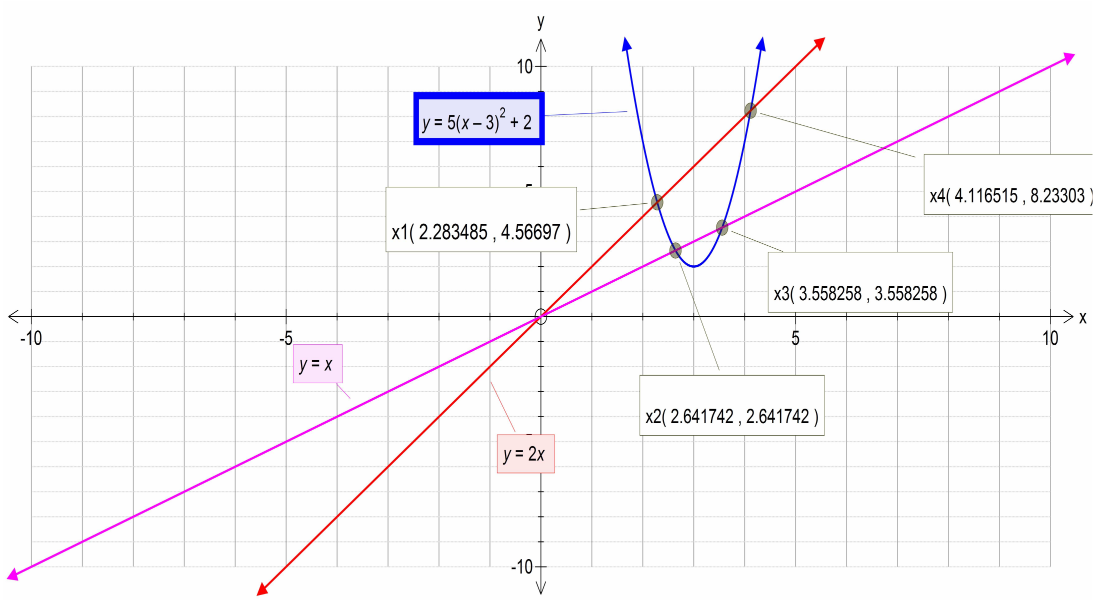
Graph 2: Value of a=2



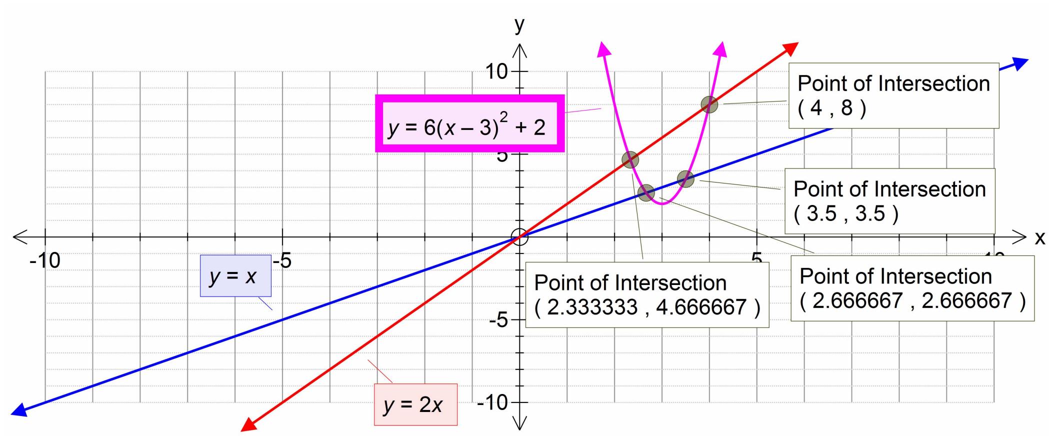
Graph 3: Value of a=3



Graph 4: Value of a=4



Graph 5 : Value of a=5



Graph 6 : Value of a=6

a	X_1	X_2	X_3	X_4	$S_L = X_2 - X_1$	$S_R = X_4 - X_3$	$D = \lvert S_L - S_R \rvert$
1	1.763932	2.381966	4.618034	6.236068	0.618034	1.618034	1
2	2	2.5	4	5	0.5	1	0.5
3	2.131483	2.565741	3.767592	4.535184	0.434258	0.767592	0.33333
4	2.219224	2.609612	3.640388	4.280776	0.390388	0.640388	0.25
5	2.283485	2.641742	3.558258	4.116515	0.358257	0.558257	0.2
6	2.333333	2.666667	3.5	4	0.333334	0.5	0.166666

Table 2: Showing results obtained from a=1 to a=6

Forming my conjecture[1]:

I can see that the value of D *is related to the value of a.* Accordingly, I can *form a hypothesis that* $D = \frac{1}{a}$

$D(1) = \frac{1}{1} = 1 \rightarrow$ TRUE

$D(2) = \frac{1}{2} = 0.5 \rightarrow$ TRUE

$D(3) = \frac{1}{3} = 0.3\overline{3} \rightarrow$ TRUE

$D(4) = \frac{1}{4} = 0.25 \rightarrow$ TRUE

$D(5) = \frac{1}{5} = 0.2 \rightarrow$ TRUE

$D(6) = \frac{1}{6} = 0.$

Proof for my conjecture[1]

I will now prove this conjecture by using mathematical deduction.

$D = \frac{1}{a}$, a>0 and $a \in Z^+$

Let the equation of any parabola be $y = ax^2 + bx + c$

Since this parabola intersects the lines y = x and y = 2x the points of Intersection of the parabola with the lines can be obtained from the Roots of the equation by equating $ax^2 + bx + c = x$ and $ax^2 + bx + c = 2x$

$ax^2 + bx + c = x$

$ax^2 + bx - x + c = 0$

$ax^2 + (b - 1)x + c = 0$

$$\text{Roots} = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$

$$\therefore x_2 = \frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a}$$

$$\therefore x_3 = \frac{-(b-1) + \sqrt{(b-1)^2 - 4ac}}{2a}$$

$S_L = x_2 - x_1$

$ax^2 + bx + c = 2x$

$ax^2 + bx - 2x + c = 0$

$ax^2 + (b - 2)x + c = 0$

$$\text{Roots} = \frac{-(b-2) \pm \sqrt{(b-2)^2 - 4ac}}{2a}$$

$$\therefore x_1 = \frac{-(b-2) - \sqrt{(b-2)^2 - 4ac}}{2a}$$

$$\therefore x_4 = \frac{-(b-2) + \sqrt{(b-2)^2 - 4ac}}{2a}$$

$S_R = x_4 - x_3$

$D = |S_L - S_R|$

$= |x_2 - x_1 - x_4 + x_3|$

$$= \left| \left(\frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a}\right) - \left(\frac{-(b-2) - \sqrt{(b-2)^2 - 4ac}}{2a}\right) - \left(\frac{-(b-2) + \sqrt{(b-2)^2 - 4ac}}{2a}\right) + \left(\frac{-(b-1) + \sqrt{(b-1)^2 - 4ac}}{2a}\right) \right|$$

$$= \left| \frac{-2}{2a} \right| \qquad = \left| \frac{-1}{a} \right| \qquad = \frac{\mathbf{1}}{|\mathbf{a}|}$$

D= $\frac{1}{a}$ since, a > 0; $|\frac{-1}{a}|$ = $\frac{1}{a}$

Hence this Conjecture is valid and has been proved for all a>0.

VALIDATION and TESTING:

Now I will test my conjecture$_1$ for all real values of 'a' in quadrant 1 starting from decimal values, then irrational numbers and then negative integers and decimals.

(Note: *F*or ALL parabolas where a $\in R^+$ I have used the eqation y=a*(x-3)²+2. B*ut for all parabolas where a $\in R^-$ I have used the eqation y= a*(x-2)²+5*. This is because when a > 0 the parabola must intersect y=*x* hence h>*k* and when a < 0 the parabola must intersect y=*2x* and hence h*<2k.)*

$y = 0.5(x - 3)^2 +$

Graph 5: Value of a=0.5

$y = 2.$

Graph 6: Value of a = 2.5

$y = 2^{0.5}(x$ -

Graph 7: Value of a=$\sqrt{2}$

$y = 10^{0.5}(x$ -

Graph 8: Value of a=$\sqrt{10}$

Graph 9: Value of a=-1

Graph 10: Value of a=-2

x2
(1

Graph 11: Value of a=-3

Point of In
(1.19098

Graph 12: Value of a=-4

$$y = -0.5(x - 2)^2 + 5$$

Graph 13: Value of a =-0.5

Graph 14: Value of a=$-\sqrt{10}$

A	X_1	X_2	X_3	X_4	S_L	S_R	D= | S_L- S_R |	D = 1/a
1	1.763932	2.381966	4.618034	6.236068	0.618034	1.618034	1	TRUE
2	2	2.5	4	5	0.5	1	0.5	TRUE
3	2.131483	2.565741	3.767592	4.535184	0.434258	0.767592	0.33333	TRUE
4	2.219224	2.609612	3.640388	4.280776	0.390388	0.640388	0.25	TRUE
0.5	1.535898	2.267949	5.732051	8.464102	0.732051	2.732051	2	TRUE
2.5	2.07335	2.536675	3.863325	4.72665	0.463325	0.863325	0.4	TRUE
$2^{0.5}$	1.882709	2.441355	4.265752	5.31505	0.558646	1.049298	0.490652	TRUE
$10^{0.5}$	2.147934	2.573967	3.742261	4.484522	0.426033	0.742261	0.316228	TRUE
-1	-0.414214	-0.302776	3.302776	2.414214	0.111438	-0.88856	1	FALSE
-2	0.5	0.633975	2.366025	3	0.133975	0.633975	0.5	FALSE
-3	0.81954	1	2.333333	2.847127	0.18046	0.513794	0.33333	FALSE
-4	1	1.190983	2.309017	2.75	0.190983	0.440983	0.25	FALSE
-0.5	-1.645751	-2.44949	2.44949	3.645751	-0.80374	1.196261	2	FALSE
$-10^{0.5}$	1.038615	0.855132	2.82864	2.328929	-0.18348	-0.49971	0.316228	FALSE

Table 3: Showing the testing for all real values of a.

<u>*Hence I can modify my conjecture*[1]*:*</u>

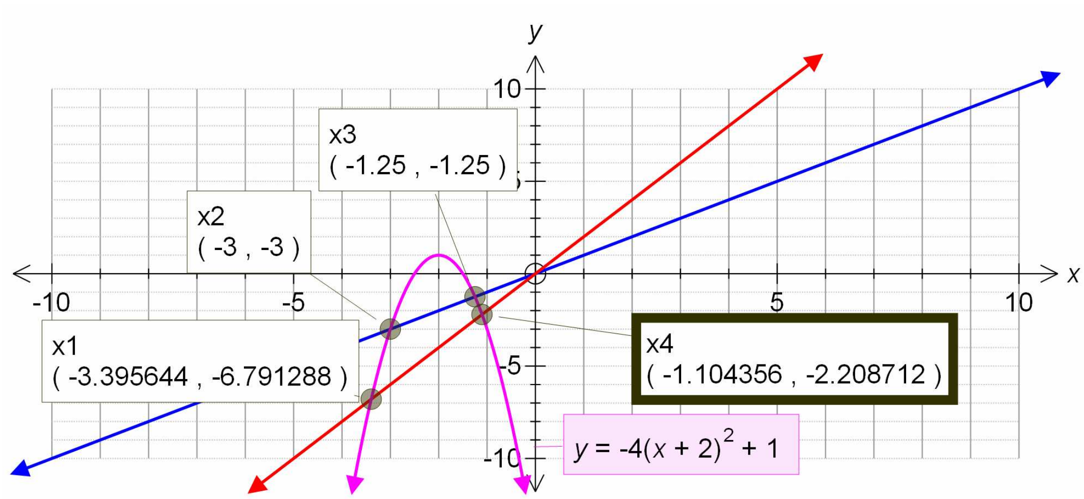
D = $\dfrac{\mathbf{1}}{|\boldsymbol{a}|}$, $a \in$ R.

- Where D is defined as $|(x_2 - x_1) - (x_4 - x_3)|$ where x_1 and x_4 are the points of intersection of lines y=2x and x_2 and x_3 are the points of intersection of lines y=x with the parabola $y=a(x-h)^2 + k$
- h and k > 0 (Vertex in the first quadrant)
- a, h and k $\in$ R

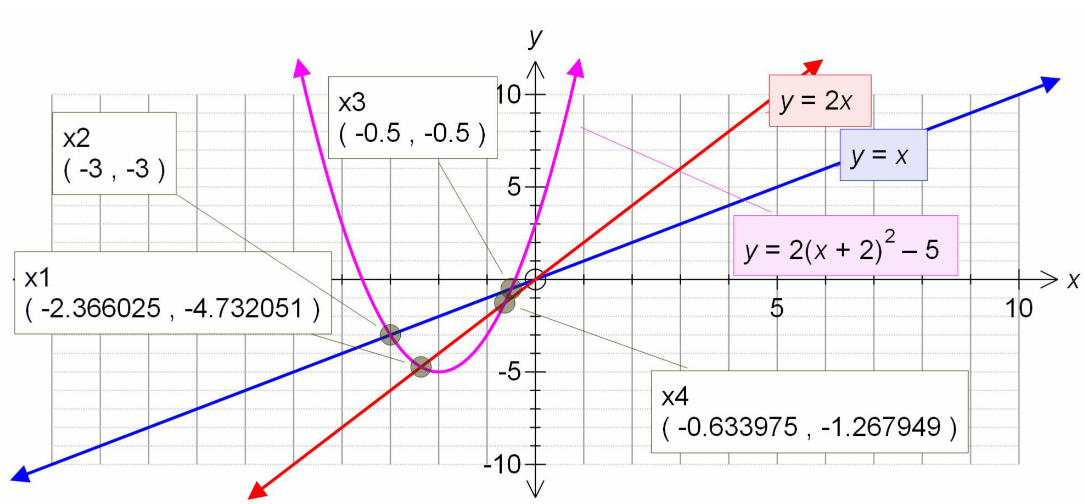
(*The proof for this modified conjecture is the same as the proof given above on page 7 where we got D =* $\dfrac{\mathbf{1}}{|\boldsymbol{a}|}$ *)*

PARABOLAS IN OTHER QUADRANTS INTERSECTING WITH LINES y=x and y=2x.

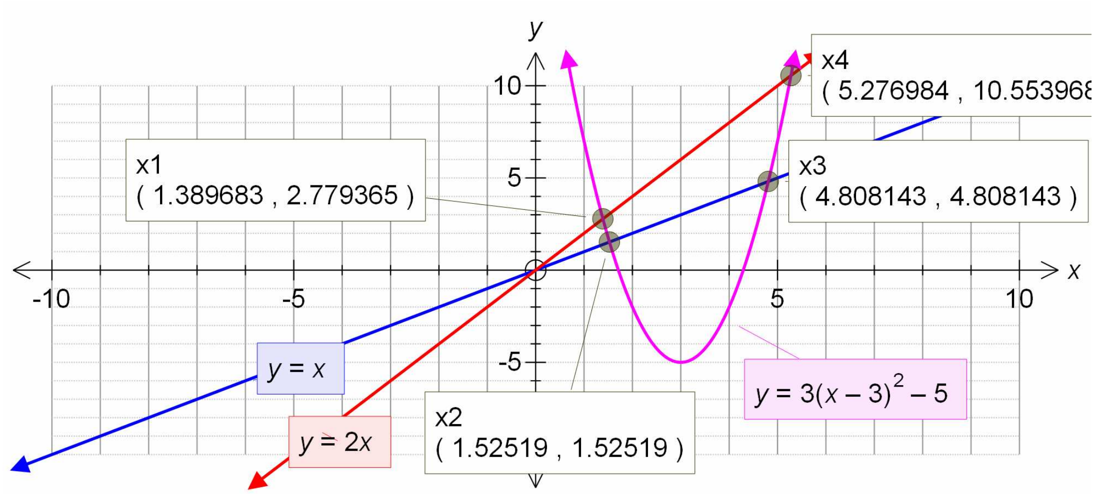
I will now see if my conjecture holds true for parabolas having their vertex in other quadrants.



Graph *15: V*alue of a = *-4* and turning point is in the 2nd quadrant.



Graph *16: V*alue of a= 2 and turning point is in the *3*rd quadrant.



Graph 17: Value of a= 3 and turning point is in the 4th quadrant.

a (quadrant #)	X_1	X_2	X_3	X_4	S_L	S_R	D=|S_L -S_R|	**D=1/|a|**
a=-4 (2nd quad)	-3.395644	-3	-1.25	-1.10436	0.395644	0.145644	0.25	**TRUE**
a=2 (3rd quad)	-2.366025	-3	-0.5	-0.63398	-0.63398	-0.13398	0.5	**TRUE**
a=3 (4th quad)	1.389683	1.52519	4.808143	5.276984	0.135507	0.468841	0.33333	**TRUE**

Table 4: Showing the testing of my conjecture for turning points in different quadrants.

Hence I *can modify my conjecture*[1]:

$$D = \frac{1}{|a|}, \quad a \in \mathbb{R}.$$

- Where D is defined as $|(x_2 - x_1) - (x_4 - x_3)|$ where x_1 and x_4 are the points of intersection of lines y=2x and x_2 and x_3 are the points of intersection of lines y=x with the parabola $y=a(x-h)^2 + k$
- a, h and k $\in$ R

(The proof for this modified conjecture is the same as the proof given above on page 7 where we got $D = \frac{1}{|a|}$ and *we know that the value of D only depends on the coefficient of* x^2 *not the values of b and c. Hence the ONLY condition for my conjecture is that the parabola MUST have four intersection points with the two lines.)*

ANY PARABOLA INTERSECTING WITH ANY 2 LINES

In the previous part of my portfolio I investigated the patterns formed in the intersection of two lines y=x and y=2x with any parabola and found D = $\frac{1}{|a|}$.

Now I will further expand my portfolio by investigating patterns formed in with any line.

Let us assume that:

- P, N are the x coefficients of straight lines y= Px + p and y=Nx + n.
- P, N $\in R$
- P < N

The basic quadratic equation y = $ax^2 + bx + c$ would intersect the two lines at four distinct

$ax^2 + bx + c = Px + p$	$ax^2 + bx + c = Nx + n$
$ax^2 + (b-P)x + (c-p) = 0$	$ax^2 + (b-N)x + (c-n) = 0$
$Roots = \frac{-(b-P) \pm \sqrt{(b-P)^2 - 4a(c-p)}}{2a}$	$Roots = \frac{-(b-N) \pm \sqrt{(b-N)^2 - 4a(c-n)}}{2a}$
$X_2 = \frac{-(b-P) - \sqrt{(b-P)^2 - 4a(c-p)}}{2a}$	$X_1 = \frac{-(b-N) - \sqrt{(b-N)^2 - 4a(c-n)}}{2a}$
$X_3 = \frac{-(b-P) + \sqrt{(b-P)^2 - 4a(c-p)}}{2a}$	$X_4 = \frac{-(b-N) + \sqrt{(b-N)^2 - 4a(c-n)}}{2a}$

points and I have found these four points, labeling the intersections with y=Px+p as X_2 and X_3 and the intersections with y=Nx+n as X_1 and X_4. Using these four points I will again find the value of D = $|\ (X_2 - X_1) - (X_4 - X_3)\ |$.

$D = |\ (X_2 - X_1) - (X_4 - X_3)\ |.$

$D = |\ X_2 - X_1 - X_4 + X_3|.$

$$D = \left| \frac{-(b-P)-\sqrt{(b-P)^2-4a(c-p)}}{2a} - \frac{-(b-N)-\sqrt{(b-N)^2-4a(c-n)}}{2a} - \frac{-(b-N)+\sqrt{(b-N)^2-4a(c-n)}}{2a} + \frac{-(b-P)+\sqrt{(b-P)^2-4a(c-p)}}{2a} \right|$$

$$D = \left| \frac{-b+P+b-N+b-N-b+P}{2a} \right|$$

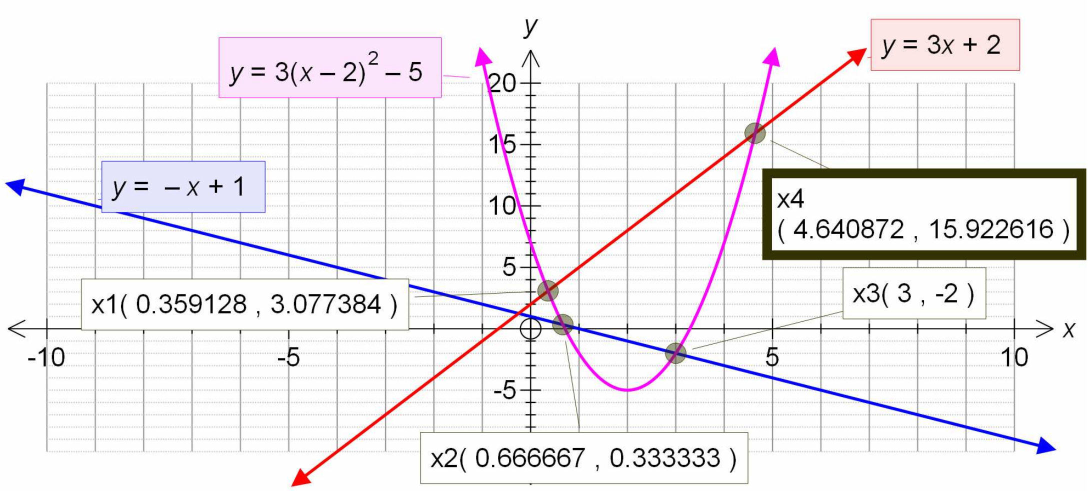
$$D = \left| \frac{2(P-N)}{2a} \right|$$

$$D = \left| \frac{(P-N)}{a} \right|$$

This is the proof by deduction for the value of D. It holds true for the conjecture[1] in the previous part of my portfolio as $D = \left|\frac{Gradiend\ of\ first\ line - Gradiend\ of\ second\ line}{a}\right| = \left|\frac{1-2}{a}\right| = \frac{1}{|a|}$

Now from this proof we can generalize that $D = \left|\frac{(P-N)}{a}\right|$, where P and N $\in R$ and are the x coefficients of 2 straight lines and a $\in$ R and is the coefficient of x^2 in a parabola. The value of D is defined as $|\ (X_2-X_1) - (X_4-X_3)\ |$ where X_1 and X_4 are the points of intersection of y=Nx +n with any parabola and X_2 and X_3 are the points of intersection of y=Px + p with the same parabola. *NOTE* here we have assumed P < N to define our 4 intersection points clearly and the value of D is dependent *ONLY* on the coefficients of x in the two lines and x^2 in the parabola.

EXAMPLE:



Graph 18: The three lines are $y=1-x$, $y=3x+2$ and $y=3(x-2)^2-5$ and are used to test my conjecture.

a	P	N	X1	X2	X3	X4	D	D= $\lvert\frac{(P-N)}{a}\rvert$
3	-1	3	0.359128	0.666667	3	4.640872	1.333333	TRUE

Table 5: Showing how D is true for any intersecting line.

$D = \lvert \frac{(-1-3)}{4} \rvert = 4/3 = 1.3333$ → Hence my conjecture is valid and true.

FOR CUBIC POLYNOMIALS

Before going on to see what happens with cubic polynomials I would like to define the value of D in other ways. We know that X_2 and X_3 are the roots of the intersection of the polynomial with the first line and X_1 and X_4 are the roots of the intersection of the polynomial with the second line. Thus:

$D = | (X_2 - X_1) - (X_4 - X_3) |$.

$D = | X_2 - X_1 - X4 + X_3|$.

$D = | X_2 + X_3 - X4 - X_1|$.

$D = | (X_2 + X_3) - (X_4 + X_1)|$.

D =| (sum of the roots of the first line) – (sum of the roots of the second line) |.

(Where the gradient of the first line is smaller than the gradient of the second)

Now for a cubic equation, we have 6 roots when it intersects with two straight lines. These six roots are X_1, X_2, X_3, X_4, X_5, and X_6.

We can express any cubic equation in its factored form as:

$$y = a(x-r_1)(x-r_2)(x-r_3), \quad \text{where } a \text{ is the coefficient of } x^3 \text{ and } r_1, r_2, \text{and } r_3 \text{ are the roots.}$$

$= a(x - r_1)(x^2 - r_2x - r_3x + r_2r_3)$

$= a(x-r_1)(x^2 - (r_2+r_3)x + r_2r_3)$

$= a(x^3 - (r_2+r_3)x^2 + r_2r_3x - r_1x^2 + r_1(r_2+r_3)x - r_1r_2r_3)$

$= a(x^3 - (r_1+r_2+r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3)$

$= ax^3 - a(r_1 + r_2 + r_3)x^2 + a(r_1 r_2 + r_1 r_3 + r_2 r_3)x - a(r_1 r_2 r_3)$

If we compare this equation to the standard cubic equation $y = ax^3+bx^2+cx+d$ then:

$b = -a(r_1 + r_2 + r_3)$

$r_1 + r_2 + r_3 = \frac{-b}{a}$

Hence we can see the sum of the roots of any cubic equation $= \frac{-b}{a}$ where 'a' is the coefficient of x^3 and 'b' is the coefficient of x^2 and this is proved above by mathematical deduction.

Now for our cubic equation we have: X_2, X_3, and X_6 as the roots of the first line intersecting with the curve and X_1, X_4, and X_5, as the roots of the second line intersecting with the curve.

$\boldsymbol{D} = |$ (sum of the roots of the first line) – (sum of the roots of the second line) $|$

$\therefore \boldsymbol{D} = |\, (X_1 + X_4 + X_5) - (X_2 + X_3 + X_6) \,|$

We know that $(X_1 + X_4 + X_5) = \frac{-b}{a}$. and $(X_2 + X_3 + X_6) = \frac{-b}{a}$.

$\therefore \boldsymbol{D} = |\frac{-b}{a} - (\frac{-b}{a})\,|$

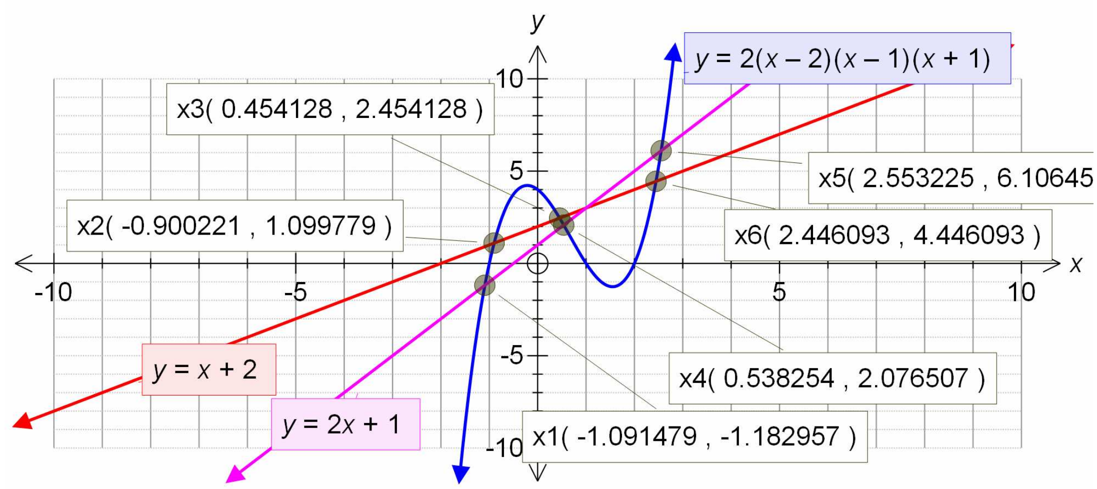
$= \frac{-b}{a} + \frac{b}{a}$

$=0$

Thus D = 0.

We know this is true as when a straight line $y= Nx+n$ intersects with a cubic curve $y= ax^3+bx^2+cx+d$ only the value of 'c' and'd' are changed. Hence the value of D is zero for any cubic polynomial as the intersecting lines do not have an influence on the value of 'a' and 'b' that determine D.

<u>EXAMPLE:</u>



Graph 19 and Table6: Showing how D = 0 for a cubic polynomial.

x1	**x2**	**x3**	**x4**	**x5**	**x6**	**D=|(x1+x4+x5)-(x2+x3+x6)|**
-1.09148	- 0.90022	0.454128	0.538254	2.553225	2.446093	**0**

Table 7: Showing how the value of D = 0 for a cubic polynomial.

FOR HIGER ORDER POLYNOMIALS:

- We can express any polynomial of degree 'n' in its factored form as:

  y $= a[(x-r_1)(x-r_2)(x-r_3)\ldots.(x-r_{n-1})(x-r_n)]$

  Where 'a' is the coefficient of x^n and r_1, r_2, r_3, …r_{n-1}, and r_n are all roots of the polynomial.

- This can also be expressed as *(observed from the cubic equation on page 18)*:

  $f(x) = ax^n - a\ (r_1 + r_2 + r_3 \ldots r_n)\ x^{n-1} + \ldots. + (-1)^n\ a\ (r_1 r_2 r_3 \ldots r_n)$

I am now going to use mathematical induction and try and prove that the sum of the roots of a polynomial of degree n is $\frac{-b}{a}$

$p(3)$ *statement is true as shown on page* 18.

$p(k)$ *statement*:

$a[(x-r_1)(x-r_2)\dots(x-r_k)] = a[x^k - (r_1 + r_2 + r_3 \dots r^k)\, x^{k-1} + \cdots . + (-1)^k\, (r_1 r_2 r_3 \dots r_k)]$

Let us assume that p (k) statement is true.

$p(k+1) statement$

Now if we multiply both sides by $(x - r_{k+1})$ →

$a[(x-r_1)(x-r_2)\dots(x-r_k)](x-r_{k+1}) = a[x^k - (r_1 + r_2 + r_3 \dots r^k)\, x^{k-1} + \cdots . + (-1)^k\, (r_1 r_2 r_3 \dots r_k)](x-r_{k+1})$

$a[(x-r_1)(x-r_2)\dots(x-r_k)](x-r_{k+1}) = a[x^{k+1} - (r_1 + r_2 + r_3 \dots r^k)\, x^{k} + \cdots . + (-1)^k\, (r_1 r_2 r_3 \dots r_k)(r_{k+1})]$

$a[(x-r_1)(x-r_2)\dots(x-r_k)(x-r_{k+1})] = a[x^{k+1} - (r_1 + r_2 + r_3 \dots r^k)\, x^{k} + \cdots . + (-1)^{k+1}\, (r_1 r_2 r_3 \dots r_k r_{k+1}]$

Hence p (k+1) statement is true.

∴ p(3),p(k) and p(k+1) statement is true and hence we have proved that the sum of the roots of a polynomial with degree 'n' is $\frac{-b}{a}$ where 'a' is the coefficient of x^n and b is the coefficient of x^{n-1}.

Now that we know that the sum of the roots of any polynomial is $\frac{-b}{a}$ we can find the value of D for any polynomial.

We already know that value of D for a parabola intersecting with two lines and for a cubic curve intersecting with two lines.

For a polynomial higher than that of degree 3, the coefficients of X^n and x^{n-1} always remain 'a' and 'b' respectively. The two lines y=Px+p and y=Nx+n do not have an effect on the value of 'a' and 'b.

Thus the value of D is →

D=| (sum of the roots of the first line) – (sum of the roots of the second line) |

We know that (sum of the roots of first line) = $\frac{-b}{a}$. and (sum of the roots of the second line) = $\frac{-b}{a}$.

$\therefore \boldsymbol{D} = |\frac{-b}{a} - (\frac{-b}{a})|$

$= \frac{-b}{a} + \frac{b}{a}$

=0

Thus D = 0 for any polynomial higher than order 2.

CONCLUSION

I can conclude my portfolio by saying that there was a pattern found with the intersection of 2 lines and a polynomial. The value of D which is defined as the modulus of the sum of the roots of the first intersecting line minus the sum of the roots of the second intersecting line was found for each polynomial.

For Polynomial of degree 2 → D= $|\frac{(P-N)}{a}|$, where P and N $\in R$ and are the x coefficients of any 2 straight lines and a $\in$ R and is the coefficient of x^2 in a parabola.

For a Polynomial of degree 3 or higher → D = 0.

These are the patterns that I found and proved in my portfolio. This portfolio has further scope and the intersections of a cubic curve with a parabola can also give interesting patterns.