1. Let $P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.
Using the sums of $(a_5 + a_3 + a_1)$ and $(a_4 + a_2 + a_0)$, determine whether $P(1) = 0$? and $P(-1) = 0$?

Examine these examples:

(1) $P(x) = x^5 - 3x^4 + 2x^3 + 4x^2 + 6x - 10$
(2) $P(x) = x^5 - 3x^4 + 2x^3 - 4x^2 + 6x + 10$
(3) $P(x) = x^5 + 3x^4 + 2x^3 - 4x^2 + 6x + 10$
(4) $P(x) = x^5 + 3x^4 + 2x^3 - 4x^2 + 6x - 10$

What is your conclusion for the general case when

$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x^1 + a_0$?

Solution:

(1) $P(x) = x^5 - 3x^4 + 2x^3 + 4x^2 + 6x - 10$
$(a_5 + a_3 + a_1) = 9$
$(a_4 + a_2 + a_0) = -9$
$P(1) = 0$
$P(-1) = -18$

(2) $P(x) = x^5 - 3x^4 + 2x^3 - 4x^2 + 6x + 10$
$(a_5 + a_3 + a_1) = 9$
$(a_4 + a_2 + a_0) = 3$
$P(1) = 12$
$P(-1) = -6$

(3) $P(x) = x^5 + 3x^4 + 2x^3 - 4x^2 + 6x + 10$
$(a_5 + a_3 + a_1) = 9$
$(a_4 + a_2 + a_0) = 9$
$P(1) = 18$
$P(-1) = 0$

(4) $P(x) = x^5 + 3x^4 + 2x^3 - 4x^2 + 6x - 10$
$(a_5 + a_3 + a_1) = 9$
$(a_4 + a_2 + a_0) = -11$
$P(1) = -2$
$P(-1) = -20$

Conclusion:

From the above examples, we see that when:
$a_5 + a_3 + a_1 = -(a_4 + a_2 + a_0)$ $\quad$ $P(1)= 0$

$a_5 + a_3 + a_1 = a_4 + a_2 + a_0$ $P(-1)= 0$

Therefore for the general case,

if $a_{n-1} + a_{n-3} + \ldots + a_1 = -(a_n + \ldots + a_2 + a_0)$ then $P(1) = 0$

if $a_{n-1} + a_{n-3} + \cdots + a_1 = an + \cdots + a_2 + a_0$ then $P(-1) = 0$

2. **There is a conclusion that states:**
   If an integer k is a zero of a polynomial with integral coefficients, then k must be a factor of the constant term of the polynomial.

To understand this conclusion, study the function $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and suppose that $P(k) = 0$. Can you see that k must be a factor of a_0?

Solution:

$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$
$P(k) = a_3k^3 + a_2k^2 + a_1k + a_0$

Suppose $P(k) = 0$ then
$a_3k^3 + a_2k^2 + a_1k + a_0 = 0$
$a_0 = -(a_3k^3 + a_2k^2 + a_1k)$
$= k\,(-a_3k^2 - a_2k - a_1)$

Therefore, k must be a factor of a_0.

3. There is a conclusion that states:

If $a_0 = 0$, then $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ has integral coefficients, and the rational number k/m in lowest terms is a zero of P(x), then k must be a factor of a_0, and m must be a factor of a_n.

To understand this conclusion, study the function $P(x)= a_3x^3+a_2x^2+a_1x+a_0$, and suppose that $P(k/m) = 0$. Can you see that k must be a factor of a_0, and m must be a factor of a_3?

Solution:

$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$
$P(k/m) = a_3(k/m)^3 + a_2(k/m)^2 + a_1(k/m) + a_0$

Since $P(k/m) = 0$
$P(k/m) = a_3(k/m)^3 + a_2(k/m)^2 + a_1(k/m) + a_0 = 0$
$= a_3k^3/m^3 + a_2k^2/m^2 + a_1k/m + a_0 = 0$
$a_0 = - (a_3k^3/m^3 + a_2k^2/m^2 + a_1k/m)$
$= k (a_3k^2/m^3 - a_2k/m^2 - a_1/m)$

Therefore, k is a factor of a_0.

$P(k/m) = a_3k^3/m^3 + a_2k^2/m^2 + a_1k/m + a_0 = 0$
$a_3 = - (a_2k^2/m^2 + a_1k/m + a_0)/ (k^3/m^{3)}$
$= m (-a_2k^2/m^3 - a_1k/m^2 - a_0/m) / (k^3/m^{3)}$

Therefore, m is a factor of a_3.

4. a-bi is called the conjugate of a+bi, where a and b are real numbers and i = (√-1). Let $\overline{a+bi}$ denote a-bi. In other words, $\overline{a+bi}$ = a-bi.

Prove that:

(1) $\overline{(a + bi) + (c + di)} = \overline{a + bi} + \overline{c + di}$
(2) $\overline{(a + bi) - (c + di)} = \overline{a + bi} - \overline{c + di}$
(3) $\overline{(a + bi)(c + di)} = \overline{(a + bi)}\ \overline{(c + di)}$
(4) $\overline{(a + bi)^3} = (\overline{a + bi})^3$

If a is a real number, $\overline{a}$ = ? $\overline{0}$ = ?

Solution:

Let $\overline{a + bi} = a - bi$

(1) $\overline{(a + bi) + (c + di)} = \overline{a + bi} + \overline{c + di}$
$\overline{(a + c) + (bi + di)} = a - bi + c - di$
$(a + c) + (b + d)i = (a + c) - (bi + di)$
$(a + c) - (b + d)i = (a + c) - (b + d)i$

(2) $\overline{(a + bi) - (c + di)} = \overline{a + bi} - \overline{c + di}$
$(a - c) + (bi + di) = (a - bi) - (c - di)$
$(a - c) - (b + d)i = (a - c) - (bi - di)$

$(a - c) - (b + d)i = (a - c) - (b - d)i$

(3) $\overline{(a + bi)(c + di)} = \overline{(a + bi)}\,\overline{(c + di)}$

$\overline{ac + adi + bci - bd} = (a - bi)(c - di)$

$\overline{(ac - bd) + (ad + bc)i} = ac - adi - bci - bd$

$(ac - bd) - (ad + bc)i = (ac - bd) - (ad + bd)i$

(4) $\overline{(a + bi)^3} = \overline{(a + bi)}^3$

$\overline{(a + bi)(a^2 + 2abi - b^2)} = (a - bi)^3$

$\overline{a^3 + 3a^2bi - 3ab^2 - b^3i} = (a - bi)(a^2 - 2abi - b^2)$

$\overline{(a^3 - 3ab^2) + (3a^2bi + b^3i)} = a^3 - 3a^2bi - 3ab^2 + b^3i$

$a^3 - 3ab^2 - 3a^2bi + b^3i = a^3 - 3ab^2 - 3a^2bi + b^3i$

Therefore, if a is a real number, $\bar{a} = \overline{a + 0i} = a - 0i = a$

Because a is real and $\bar{a} = a$, 0 is real

Therefore, $0 = 0$

5. There is a conclusion that states:

If $a + bi$ is a zero of $P(x)= a_3x^3 + a_2x^2 + a_1x + a_0$, where a, b, a_3, a_2, a_1, and a_0 are real numbers, then its conjugate, $a - bi$, is also a zero of $P(x)$.

Use the results found in question 4 to prove this conclusion. And then, state the general conclusion if $P(x)= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$.

Solution:

From the results in question 4, we know $\bar{a} = a$.

So $P(x)= a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

$= \overline{a_3x^3 + a_2x^2 + a_1x + a_0} = \bar{0} = 0$

$P(a + bi) = \overline{a_3(a + bi)^3 + a_2(a + bi)^2 + a_1(a + bi) + a_0} = 0$

$= a_3\,\overline{(a + bi)}^3 + a_2\,\overline{(a + bi)}^2 + a_1\,\overline{(a + bi)} + a_0 = 0$

$= a_3\,(a - bi)^3 + a_2\,(a - bi)^2 + a_1\,(a - bi) + a_0 = 0$

$\overline{(a + bi)3} = \overline{(a + bi)}3$

$\overline{(a + bi)} = a - bi$

Therefore a – bi is a zero of P(x)

Conclusion:
If a complex number, a + bi, is a zero of a polynomial function

$P(x)= a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, with real coefficients, then its conjugate,

a – bi, is also a zero of the polynomial.

6. **Let us call a - b √ r the conjugate radical of a + b √ r, where a, b and r are rational numbers and √ r is irrational. And let $\overline{a + b\sqrt{r}}$ denote a - b √ r, that is, $\overline{a + b\sqrt{r}} = a - b\sqrt{r}$.**

   Prove that:
   (1) $\overline{(a + b\sqrt{r}) + (c + d\sqrt{r})} = \overline{a + b\sqrt{r}} + \overline{c + d\sqrt{r}}$
   (2) $\overline{(a + b\sqrt{r}) - (c + d\sqrt{r})} = \overline{a + b\sqrt{r}} - \overline{c + d\sqrt{r}}$
   (3) $\overline{(a + b\sqrt{r})(c + d\sqrt{r})} = \overline{(a + b\sqrt{r})}\;\overline{(c + d\sqrt{r})}$
   (4) $\overline{(a + b\sqrt{r})^3} = (\overline{a + b\sqrt{r}})^3$

If a is a rational number, $\overline{a}$ = ? $\overline{0}$ = ?

Solution:

(1) $\overline{(a + b\sqrt{r}) + (c + d\sqrt{r})} = \overline{a + b\sqrt{r}} + \overline{c + d\sqrt{r}}$

$(a + c) + (b\sqrt{r} + d\sqrt{r}) = (a - b\sqrt{r}) + (c - d\sqrt{r})$

$(a + c) - (b\sqrt{r} + d\sqrt{r}) = (a + c) - (b\sqrt{r} + d\sqrt{r})$

(2) $\overline{(a + b\sqrt{r}) - (c + d\sqrt{r})} = \overline{a + b\sqrt{r}} - \overline{c + d\sqrt{r}}$

$a + b\sqrt{r} - c - d\sqrt{r} = (a - b\sqrt{r}) - (c - d\sqrt{r})$

$(a - c) + (b - d)\sqrt{r} = (a - c) - (b\sqrt{r} - d\sqrt{r})$

$(a - c) - (b - d)\sqrt{r} = (a - c) - (b - d)\sqrt{r}$

(3) $\overline{(a + b\sqrt{r})(c + d\sqrt{r})} = \overline{(a + b\sqrt{r})}\,\overline{(c + d\sqrt{r})}$

$ac + ad\sqrt{r} + bc\sqrt{r} + bdr = (a - b\sqrt{r})(c - d\sqrt{r})$

$(ac + bdr) + (ad\sqrt{r} + bc\sqrt{r}) = ac - ad\sqrt{r} - bc\sqrt{r} + bdr$

$(ac + bdr) - (ad\sqrt{r} + bc\sqrt{r}) = ac + bdr - ad\sqrt{r} - bc\sqrt{r}$

$ac - ad\sqrt{r} - bc\sqrt{r} + bdr = ac - ad\sqrt{r} - bc\sqrt{r} + bdr$

(4) $\overline{(a + b\sqrt{r})^3} = \overline{(a + b\sqrt{r})}^3$

$(a + b\sqrt{r})(a^2 + 2ab\sqrt{r} + b^2r) = (a - b\sqrt{r})^3$

$a^3 + 3a^2b\sqrt{r} + 3ab^2r + b^3r\sqrt{r} = (a - b\sqrt{r})(a^2 - 2ab\sqrt{r} + b^2r)$

$(a^3 + 3ab^2r) + (3a^2b\sqrt{r} + b^3r\sqrt{r}) = a^3 - 3a^2b\sqrt{r} + 3ab^2r - b^3r\sqrt{r}$

$(a^3 + 3ab^2r) - (3a^2b\sqrt{r} + b^3r\sqrt{r}) = (a^3 + 3ab^2r) - (3a^2b\sqrt{r} + b^3r\sqrt{r})$

Because a is real and $\overline{a}$ = a, 0 is real
Therefore, 0 = 0

7. There is a conclusion that states:

If $a + b\sqrt{r}$ is a zero of $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, where a, b, r, a_3, a_2, a_1, and a_0 are rational numbers, but $\sqrt{r}$ is irrational, then its conjugate radical, $a - b\sqrt{r}$ is also a zero of P(x).

Use the results found in question 6 and prove this conclusion.

And then, state what general conclusion should we deal with if

$P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$.

Solution:

From the results in question 6, we know $\bar{a} = a$

Therefore

$$P(x) = \underline{a_3x^3 + a_2x^2 + a_1x + a_0} = \underline{0}$$
$$= a_3x^3 + a_2x^2 + a_1x + a_0 = 0 = 0$$

$$P(a + b\sqrt{r}) = \overline{\underline{a_3}\,\underline{(a + b\sqrt{r})}^3 + \underline{a_2}\,\underline{(a + b\sqrt{r})}^2 + \underline{a_1}\,\underline{(a + b\sqrt{r})} + \underline{a_0}} = 0$$
$$= a_3 (a + b\sqrt{r})^3 + a_2 (a + b\sqrt{r})^2 + a_1 (a + b\sqrt{r}) + \underline{a_0} = 0$$

$$\overline{(a + b\sqrt{r})^3} = (\overline{a + b\sqrt{r}})^3$$

$$= a_3 (a - b\sqrt{r})^3 + a_2 (a - b\sqrt{r})^2 + a_1 (a - b\sqrt{r}) + a_0 = 0$$

$$\overline{a + b\sqrt{r}} = a - b\sqrt{r}$$

Therefore, $a - b\sqrt{r}$ is a zero of P(x).

Conclusion:

If $a + b\sqrt{r}$ is a zero of a polynomial function $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, with rational coefficients, where a and b are rational, but $\sqrt{r}$ is irrational, then the conjugate radical, $a - b\sqrt{r}$, is also a zero of the polynomial.

8. There is a conclusion that states:

The sum of the zeros of the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, with $a_n \neq 0$, is equal to $-a_{n-1} / a_n$, and the product of the zeros is equal to a_0 / a_n if n is even and $- a_0 / a_n$ if n is odd.

To understand this conclusion, study the function $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, with $a_3 \neq 0$, and suppose that x_1, x_2, and x_3 are its three zeros. Can you see that $x_1 + x_2 + x_3 = -a_2 / a_3$, and $x_1x_2x_3 = -a_0 / a_3$?

Solution:

If $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $a_3 \neq 0$, its zeros are x_1, x_2, x_3.

$P(x) = x^3 + (a_2 / a_3)x^2 + (a_1 / a_3)x + (a_0 / a_3) = 0.$ (divide by a_3)

$$x^3 + (a_2 / a_3)x^2 + (a_1 / a_3)x + (a_0 / a_3) = (x - x_1)(x - x_2)(x - x_3)$$
$$= x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_2x_3 + x_3x_1)x - x_1x_2x_3$$

Therefore, $x_1 + x_2 + x_3 = -a_2 / a_3$ and $x_1x_2x_3 = -a_0 / a_3.$

Conclusion:

In the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, with $a_n \neq 0$, the sum of the zeros is equal to $-a_{n-1} / a_n$, and the product of the zeros is equal to a_0 / a_n if n is even and $- a_0 / a_n$ if n is odd.