1 Sample project

This Maths Studies project has been graded by a moderator.
As you read through it, you will see comments from the moderator in boxes like this:

Moderator's comment:

At the end of the sample project is a summary of the moderator's grades, showing how the project has been graded against all the criteria A to G. These criteria are explained in detail in chapter 13 of the Mathematical Studies textbook.

Reading projects and the moderator's comments will help you to see where marks are gained and lost, and will give you helpful tips in writing your own project.

Investigating the surface areas of 1 litre drink packaging of milk, orange juice and water

Contents

Aim:

The aim of the project is to investigate the ideal packaging surface area for 1 litre of milk, orange juice and water.

Introduction:

Most of the common packages of drinks found in fridges, such as milk, orange juice and water, come in different shaped packages but each have a volume of 1 litre. In this investigation I am going to calculate the ideal shape to hold a volume of 1 litre. I chose milk, orange juice and water as all three have different shaped packaging with different sized faces. I will use calculus to find the dimensions for the least surface area for all three packages. I will then calculate the actual surface area and the minimum surface area and find the percentage difference to see which of the packages is closest to the minimum. I will also draw the graph for the equation of the surface area using Autograph and find the minimum value from the graph to validate my calculations.

Moderator's comment:
The project has a title, a task and a plan that is not detailed.

Measurements:

I will measure, in cm, the width, length and height of the orange juice and milk carton and the base radius and height of the water bottle.

Orange Juice

Width = 6 cm

Length = 9 cm

Height = 19.5 cm

Milk

Width = 7 cm

Length = 7 cm

Height = 19.5 cm

Water

Radius = 3.75 cm

Height = 23.5 cm

This is the bottle of water. I measured the radius and height and ignored the shape at the top and bottom and assumed that it was a cylinder.

I will use x to represent the radius of the circle and y to represent the height of the bottle.

Using the formula for the volume of a cylinder

$V = \pi x^2 y$

I was able to calculate the volume of water in the bottle.

$V = \pi(3.75)^2 \times 23.5 = 1038 \text{ cm}^3 = 1038 \text{ ml}$

Moderator's comment: Simple mathematics

This is 38 ml more than is stated on the bottle but this is probably due to making the bottle a perfect cylinder which it is not.

The surface area, S, of a cylinder is given by the formula:

$$S = 2\pi x^2 + 2\pi xy$$

Using the formula for the volume of a cylinder and the fact that the bottle contains 1000 ml we get $1000 = \pi x^2 y$

Rearranging this formula for y gives $y = \frac{1000}{\pi x^2}$

Substituting for y into the surface area formula:

We get $$S = 2\pi x^2 + 2\pi x\left(\frac{1000}{\pi x^2}\right)$$
$$= 2\pi x^2 + 2000x^{-1}$$

Moderator's comment: Another simple process.

I will differentiate S with respect to x and then equate this expression to zero in order to find the maximum or minimum value for x.

$\frac{dS}{dx} = 4\pi x - 2000x^{-2} = 0$ at maximum and minimum values

Moderator's comment: Further process

Multiplying by x^2 gives:

$\Rightarrow 4\pi x^3 - 2000 = 0$

$\Rightarrow x^3 = \frac{2000}{4\pi} = 159.15$

$\Rightarrow x = 5.42 \text{ cm}$

$\Rightarrow y = 10.84 \text{ cm}$

This means that a base radius of 5.42 cm and a height of 10.84 cm will produce the minimum surface area of the bottle of water.

So, the minimum surface area is $S = 2\pi(5.42)^2 + 2\pi(5.42)(10.84)$

$= 554 \text{ cm}^2$

The actual surface area is $S = 2\pi(3.75)^2 + 2\pi(3.75)(23.5) = 642 \text{ cm}^2$

The difference is 88 cm^2

So, the percentage difference is $\frac{88}{642} \times 100 = 13.7\%$

Moderator's comment: Simple process

These measurements would form a cylinder where the diameter of the base was equal to the height. This would be very awkward to hold and would not fit in a refrigerator door. This implies that the dimensions that give the minimum surface area are not always the best. The company has to take other things into account such as ergonomics and practicalities.

This is the orange juice package. It is a cuboid. The length of the rectangular base is 1.5 times its width. I will use x to represent the width of the base and y to represent the height of the package.

So the formula for Volume, V, is:

$V = x(1.5x)y$

The actual volume is $V = 6 \times 9 \times 19.5 = 1053$ cm³

This is 53 ml more than is stated on the package but the package is probably not completely full.

The surface area, S, for a cuboid is given by the formula:

$S = 2x(1.5x) + 2xy + 2(15x)y$

Using the fact that the actual volume of the package is 1000 ml, I get

$1000 = 1.5x^2y$

Rearranging this formula for y I get

$y = \frac{1000}{1.5x^2}$

Substituting this expression for y into the equation for the surface area, I get:

$S = 3x^2 + 2x \times \frac{1000}{1.5x^2} + 3x \times \frac{1000}{1.5x^2}$

$S = 3x^2 + 1333.3x^{-1} + 2000x^{-1}$

Differentiating S with respect to x

$\frac{dS}{dx} = 6x - 1333.3x^{-2} - 2000x^{-2} = 0$ at maximum and minimum values

Multiplying by x^2

⇨ $6x^3 - 3333.3 = 0$

⇨ $x^3 = 555.55$

⇨ $x = 8.22$ cm

⇨ $y = 9.87$ cm

So, the best dimensions are 8.22 cm by 12.33 cm by 9.87 cm

This gives a surface area of:

$S = 2(1.5)(8.22)^2 + 2(8.22)(9.87) + 2(1.5)(8.22)(9.87)$

$= 608$ cm²

The actual surface area is $S = 2(6)(9) + 2(6)(19.5) + 2(9)(19.5)$

$= 693$ cm²

The difference = 85 cm²

Percentage difference = $\frac{85}{693} \times 100 = 12.3\%$

This is slightly less that for the bottle of water.

These dimensions would give a cuboid where the width was larger than the height. Once again this would not be very practical for everyday use as it would be difficult to hold and to store.

This is the milk carton. I am ignoring the top and taking the shape to be a cuboid with width x and height y. The base is a square.

The volume is $V = x^2y$

The actual volume, V, is

$V = 7^2(19.5) = 955.5\ \text{cm}^3$

This is less than the amount stated on the package. However, the carton was bulging – so I will assume that it did contain 1 litre.

The formula for the surface area, S, of this cuboid is

$S = 2x^2 + 4xy$

Using the fact that the milk carton contains 1000 ml we get

$1000 = x^2y$

Rearranging the formula for the volume for y gives:

$$y = \frac{1000}{x^2}$$

Substituting this expression into the formula for the surface area gives:

$S = 2x^2 + 4x \times \frac{1000}{x^2}$

$S = 2x^2 + 4000x^{-1}$

Differentiating S with respect to x gives

$\frac{dS}{dx} = 4x - 4000x^{-2} = 0$ at maximum or minimum points

Multiplying throughout by x^2

$\Rightarrow 4x^3 - 4000 = 0$
$\Rightarrow x^3 = 1000$
$\Rightarrow x = 10\ \text{cm}$
$\Rightarrow y = 10\ \text{cm}$

So the best dimensions are 10 cm by 10 cm by 10 cm. This is a cube.

This gives a surface area $S = 2(10)^2 + 4\,(10)(10) = 600\ \text{cm}^2$

The actual surface area is $S = 2(7)^2 + 4(7)(19.5) = 644\ \text{cm}^2$

The difference is $44\ \text{cm}^2$

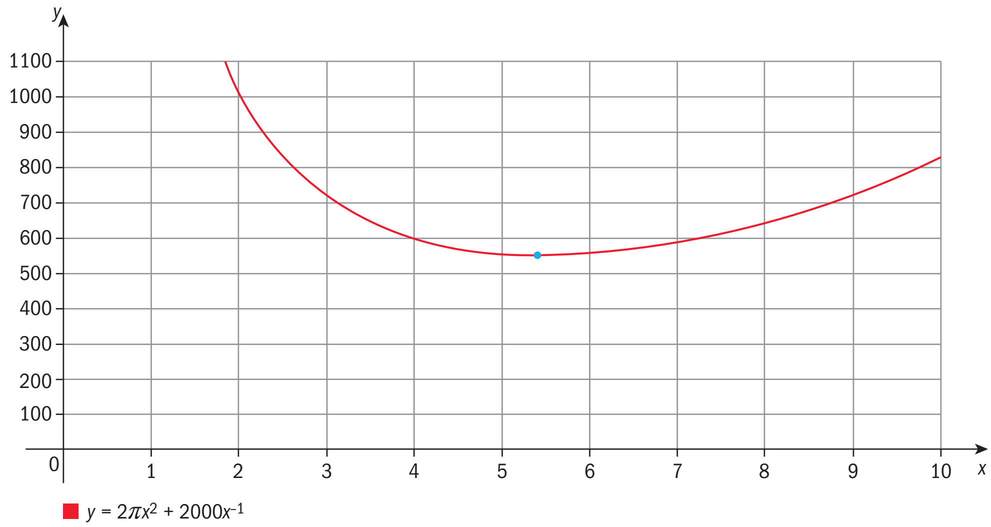
The percentage difference is $\frac{44}{644} \times 100 = 6.83\%$

So, the packaging for the milk has the smallest percentage difference and is the closest to the ideal shape and wastes the least amount of packaging.

Once again, these measurements would form a cuboid which would be difficult to hold and to store.

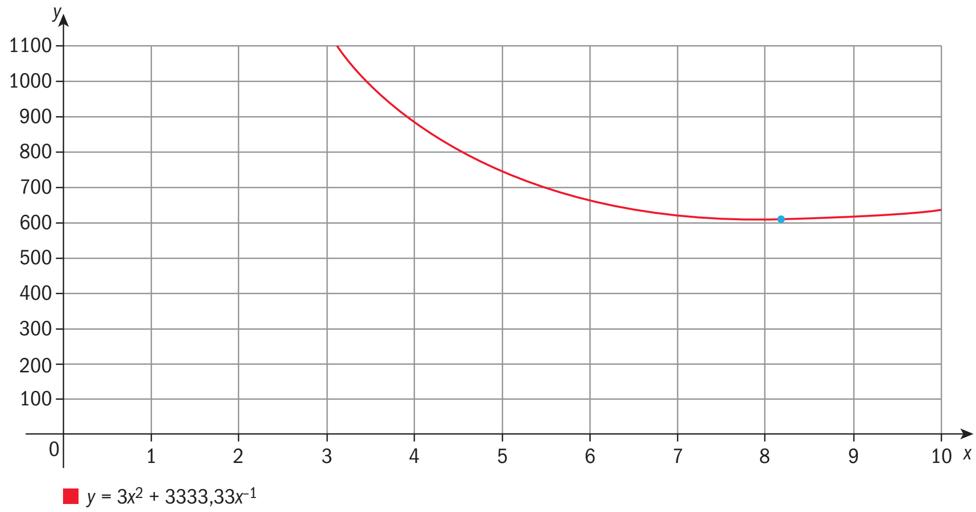
To test that these were all **minimum** values I drew the graphs on Autograph.

Water bottle



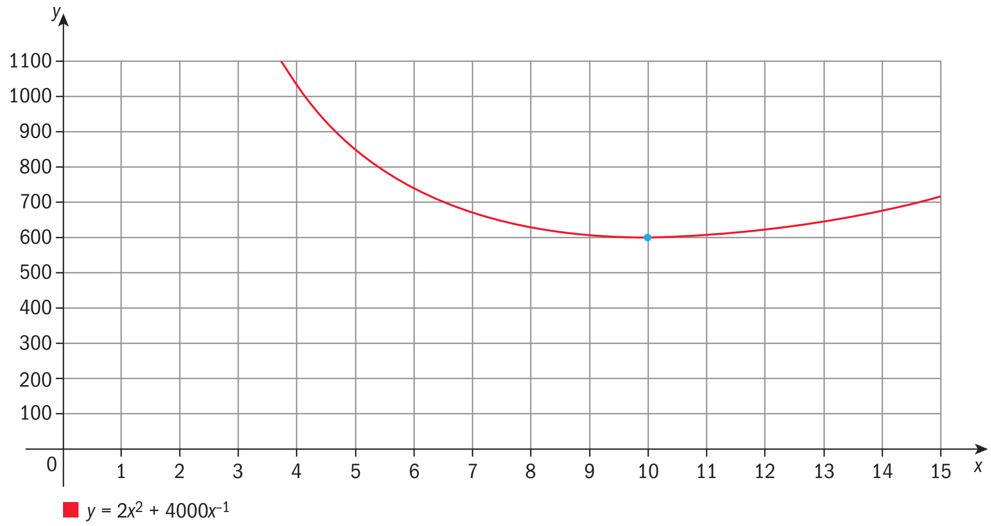
The minimum value was (5.419, 553.6) which agrees with my calculations.

Orange juice



The minimum value was (8.221, 608.2) which agrees with my calculations.

Milk Carton



The minimum value was (10, 600) which agrees with my calculations.

Conclusion and validity

Looking at the three different results from these three packages, it is clear that the water was the closest to the ideal package for volume but the furthest away for the surface area. The milk carton was the closest to the ideal shape for the surface area. I am satisfied that using differentiation was an efficient process in order to find the maximum or minimum value and then sketching the graph of the expression confirmed that it was indeed a minimum value.

The shape and design of the packages affects the amount of volume the package is able to hold.

However, although I have found the minimum surface area for each of the packages, none of them are an ideal shape. The ideal shape for the milk carton is a cube, the package for the orange juice is wider than it is tall and the water bottle has the same value for diameter and height. So, none of them are the ideal shape for a few reasons:

- The packaging is not practical. Most customers would have to use both hands to pick it up.
- It would not fit into the space in the door compartment of the fridge.
- Each company has its trademark design which enables the customers to distinguish which brand product it is and, if all the packages were the same, they would lose the attraction for the customers.

Manufacturers of these packages can calculate the best size of packaging. However, they must produce the size of package which is consumer friendly, easy to mass produce, safe to transport and most profitable for the company.

In finding the formulae for the packages I ignored parts of the design and used the formula for regular shapes. This would have affected my results. Also, I only used a ruler to measure the dimensions, so this could have been a bit inaccurate and also make a difference to the final result.

Moderator's comment: The conclusions are consistent with the mathematical processes used but the student could have expanded on these quite easily. There is an attempt made to comment on the validity of the process used, the measurements and the fact that the volumes and surface areas used were not perfect shapes in reality.

Bibliography

IB Course Companion: Mathematical Studies; Bedding, Coad, Forrest, Fussey and Waldman; 08/03/2007

Summary of moderators' grades

Criterion	Grade	Comment
A	2	The project does have a title, a statement of the task and a description of the plan. However, the plan is not detailed enough to award 3 marks.
B	2	The actual measurements collected are few but sufficient for the project and they are set up for use. So 2 marks were awarded, out of a possible 3.
C	5	All the simple processes used are correct and relevant. Differentiation is a further process which is also correct and relevant. (5 out of 5 marks awarded.)
D	2	The interpretations are consistent with the processes used but the candidate could have discussed the results in much more detail. 2 marks awarded, out of a possible 3.
E	1	There is an attempt made to discuss validity. (1 out of 1 mark awarded.)
F	2	The project has been structured but it does not flow well. Hence it was decided to award 2 marks instead of 3 here.
G	2	Mathematical notation and terminology is correct throughout the project. 2 out of 2 marks awarded.
Total grade	**16**	