Mathematics Higher Level

Assignment: Mathematics Portfolio Type II:

*Creating a Logistic Model*

School: Trinity Grammar School

School Code:

Name of Student: Cassian Ho

Student Number:

Introduction

If a hydroelectric project is expected to cerate a large lake into which some fish are to be placed, a biologist estimates that if 10,000 fish were introduced into the lake, the population of fish would increase by 50% in the first year, but the long-term sustainable limit would be about 60,000.

In order to estimate the growth rate of the population of fish, it is best to find a linear growth factor for. We do this by finding two ordered pairs in the form $(u_0, r_0)$, $(u_n, r_n)$.

$r_n$ is the growth rate when the population is n.

since we know from the information given that when there are 60000 fish in the lake, the growth rate is stable i.e. 1, this can be represented as one of our ordered pairs: (60000, 1). We also know that when 10000 fish are in the lake, the growth rate is 50% i.e. population is multiplied by 1.5, so the second ordered pair is (10000, 1.5).

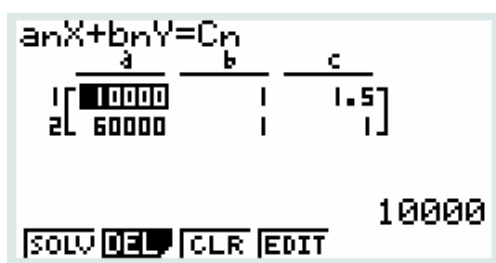from these pairs we have:

$r_{10000} = 1.5$
$r_{60000} = 1$

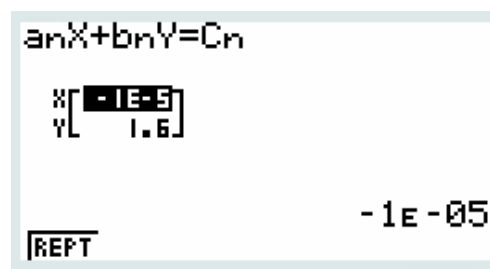and since we are trying to search for a linear function of the growth rate, we can also denote $r_n$ as:

$r_n = mn + b$    where n = population of fish. Substituting the two ordered pairs, we have:

$1.5 = m(10000) + b$
$1 = m(60000) + b$

Putting this into the GDC, we find that the solutions to the two unknowns are:



$m = -0.00001$
$b = 1.6$

from this, we can make the conjecture for the linear growth factor of fish in terms of $U_n$:
$r_n = -0.00001 \times u_n + 1.6$

Since geometric population growth models take the form: $u_{n+1} = r \times u_n$

and we also have that $r_n = -0.00001 \times u_n + 1.6$, we find that the function for $u_{n+1}$ is:

$u_{n+1} = (-0.00001 \times u_n + 1.6) \times u_n$

This is rather obvious because population size of the next year is equal to population size this year multiplied by the growth rate.

Now, let us explore what would happen if we left this population of fish to grow for 20 years, assuming that our models for total population and growth rate are correct. It is best to use Microsoft Excel 2007 to calculate this. Initially there is a population of 10000:

| Year | Population of Fish | Rate of Growth |
|------|-------------------|----------------|
| $t$ | $u_{n+1} = (-0.00001 \times u_n + 1.6) \times u_n$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 10000 | 1.5 |
| 1 | 15000 | 1.45 |
| 2 | 21750 | 1.3825 |
| 3 | 30069.375 | 1.29930625 |
| 4 | 39069.32687 | 1.209306731 |
| 5 | 47246.79997 | 1.127532 |
| 6 | 53272.27888 | 1.067277211 |
| 7 | 56856.28924 | 1.031437108 |
| 8 | 58643.68652 | 1.013563135 |
| 9 | 59439.07875 | 1.005609213 |
| 10 | 59772.48517 | 1.002275148 |
| 11 | 59908.47644 | 1.000915236 |
| 12 | 59963.30681 | 1.000366932 |
| 13 | 59985.30926 | 1.000146907 |
| 14 | 59994.12155 | 1.000058785 |
| 15 | 59997.64827 | 1.000023517 |
| 16 | 59999.05925 | 1.000009407 |
| 17 | 59999.62369 | 1.000003763 |
| 18 | 59999.84948 | 1.000001505 |
| 19 | 59999.93979 | 1.000000602 |
| 20 | 59999.97592 | 1.000000241 |

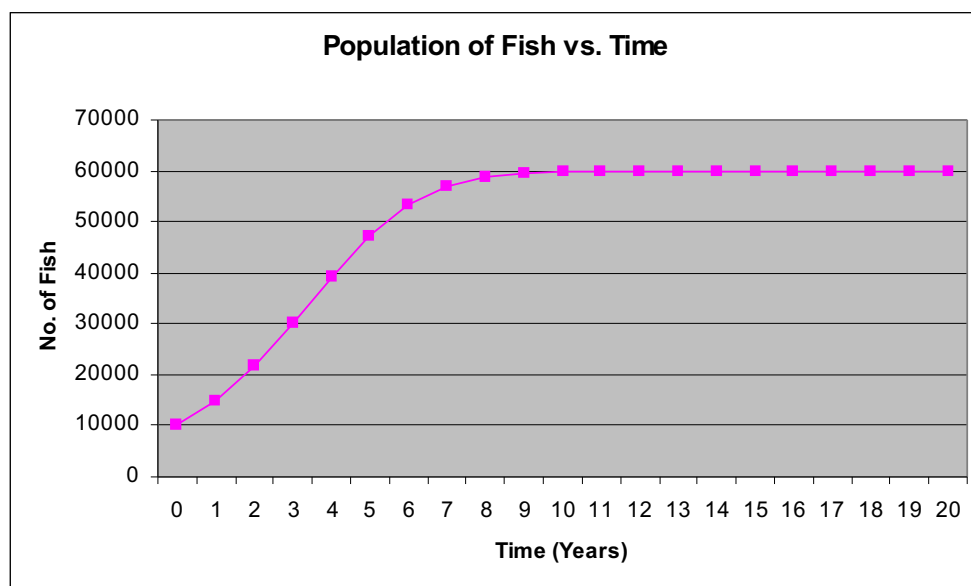(This data was achieved by using formulas in Microsoft Excel)

With these data values, it is possible to find a logistic function for the population size of fish by using the GDC. Based on the values given in the table above, the calculator is able to estimate the logistic function of $u_{n+1}$. The function given by the calculator is:

```
LogisticReg
  a =5.90726375
  b =0.61039026
  c =60406.6406
 MSe=431551.346
y=c÷(1+a·e^(-bx))

              COPY DRAW
```

However, the function that the calculator gives is just an estimate, so it is safe for us to round off this function to:

$$y = \frac{60000}{1 + 6e^{-0.6x}}$$

Putting the data obtained from the excel table into a line graph, we have:

**Population of Fish vs. Time**

No. of Fish (y-axis: 0, 10000, 20000, 30000, 40000, 50000, 60000, 70000)

Time (Years) (x-axis: 0 to 20)

This graph shows that starting from a population of 10000 fish, the rate of growth will increase initially, and then decrease so that at approximately year 10, there is a stable growth rate. This is an S-shaped curve.

Next we should consider how a different initial growth rate will affect the graph. For example, it may be the case that biologists speculate that the initial growth rate of fish may vary considerably. I will now investigate functions models for $u_{n+1}$ with growth rates r = 2, 2.3, 2.5.

Model for Growth Rate = 2

When we have a new initial growth rate, the ordered pairs i.e. ($u_n$, $r_n$) have now changed to:
(60000, 1)
(10000, 2)

To find the linear growth factor, we must form the two equations:

2 = m(10000) + b
1 = m(60000) + b



Solving this on the GDC, we have $m$ = -0.00002 and $b$ = 2.2 so the linear growth factor is:

$r_n$ = -0.00002 × $u_n$ + 2.2

and therefore the function for $u_{n+1}$ for is:
$u_{n+1}$ = (-0.00002 × $u_n$ + 2.2) $u_n$

If we let 10000 fish with an initial growth rate of 2 cultivate for 20 years, the population of growth would be:

| Year | Population of Fish | Rate of Growth |
|---|---|---|
| $t$ | $u_{n+1} = (-0.00002 × u_n + 2.2) u_n$ | $r_n = -0.00002 × u_n + 2.2$ |
| 0 | 10000 | 2 |
| 1 | 20000 | 1.8 |
| 2 | 36000 | 1.48 |
| 3 | 53280 | 1.1344 |
| 4 | 60440.832 | 0.99118336 |
| 5 | 59907.94694 | 1.001841061 |
| 6 | 60018.24114 | 0.999635177 |
| 7 | 59996.34512 | 1.000073098 |
| 8 | 60000.73071 | 0.999985386 |
| 9 | 59999.85385 | 1.000002923 |

| 10 | 60000.02923 | 0.999999415 |
| 11 | 59999.99415 | 1.000000117 |
| 12 | 60000.00117 | 0.999999977 |
| 13 | 59999.99977 | 1.000000005 |
| 14 | 60000.00005 | 0.999999999 |
| 15 | 59999.99999 | 1 |
| 16 | 60000 | 1 |
| 17 | 60000 | 1 |
| 18 | 60000 | 1 |
| 19 | 60000 | 1 |
| 20 | 60000 | 1 |

As we have done before, we should also use the GDC to find an estimate for the logistic function:

LogisticReg
  a =6.7860782
  b =1.22343431
  c =60654.3399
  MSe=2.3787E+06
y=c÷(1+a·e^(-bx))

COPY DRAW

However, the function that the calculator gives is just an estimate, so it is safe for us to round off this function to:

$$y = \frac{60000}{1 + 7e^{-1.2x}}$$

Putting this in a graph, we have:

**Population of Fish vs. Time (Initial rate = 2.0)**

No. of Fish vs. Time (Years)
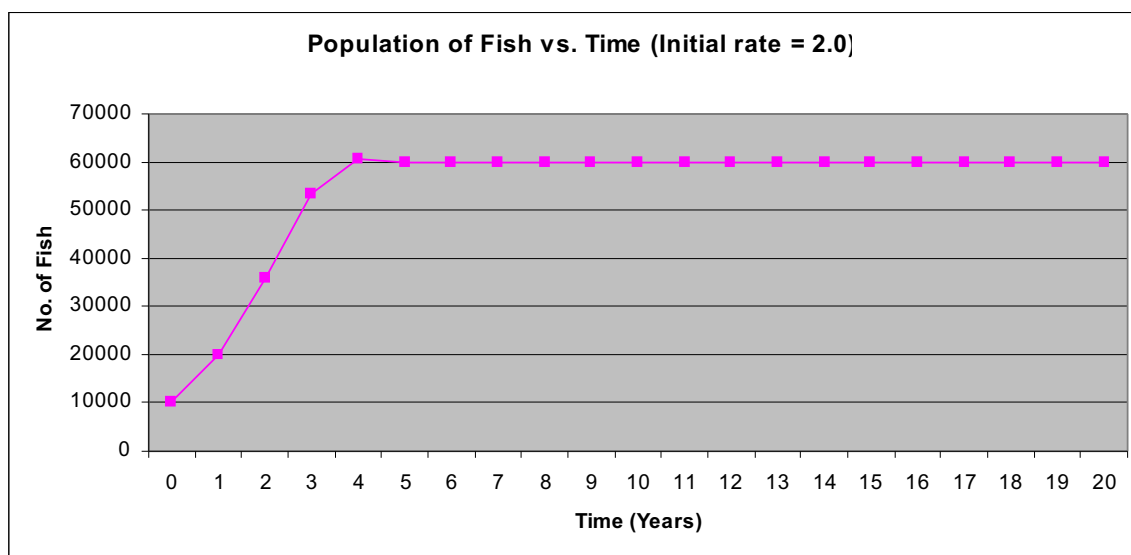
Model for Growth Rate = 2.3

When we have a new initial growth rate, the ordered pairs i.e. $(u_n, r_n)$ have now changed to:
(60000, 1)
(10000, 2.3)

To find the linear growth factor, we form the two equations:
$2.3 = m(10000) + b$
$1 = m(60000) + b$

```
anX+bnY=Cn
      a        b       c
1[ 10000    |   2.3]
2[ 60000    |    1]



           10000
SOLV DEL CLR EDIT
```

```
anX+bnY=Cn

    X[ -2E-5]
    YL  2.56]


          -2.6E-05
REPT
```

Solving this on the GDC, we have $m$ = -0.000026 and $b$ = 2.56 so the linear growth factor is:

$r_n = -0.000026 \times u_n + 2.56$

and therefore the function for $u_{n+1}$ for is:

$u_{n+1} = (-0.000026 \times u_n + 2.56)\, u_n$

If we let 10000 fish with an initial growth rate of 2.3 cultivate for 20 years, the population of growth would be:

| Year | Population of Fish | Growth Rate |
|---|---|---|
| $t$ | $u_{n+1} = (-0.000026 \times u_n + 2.56)\, u_n$ | $r_n = -0.000026 \times u_n + 2.56$ |
| 0 | 10000 | 2.3 |
| 1 | 23000 | 1.962 |
| 2 | 45126 | 1.386724 |
| 3 | 62577.30722 | 0.932990012 |
| 4 | 58384.00263 | 1.042015932 |
| 5 | 60837.06089 | 0.978236417 |
| 6 | 59513.02846 | 1.01266126 |
| 7 | 60266.53839 | 0.993070002 |

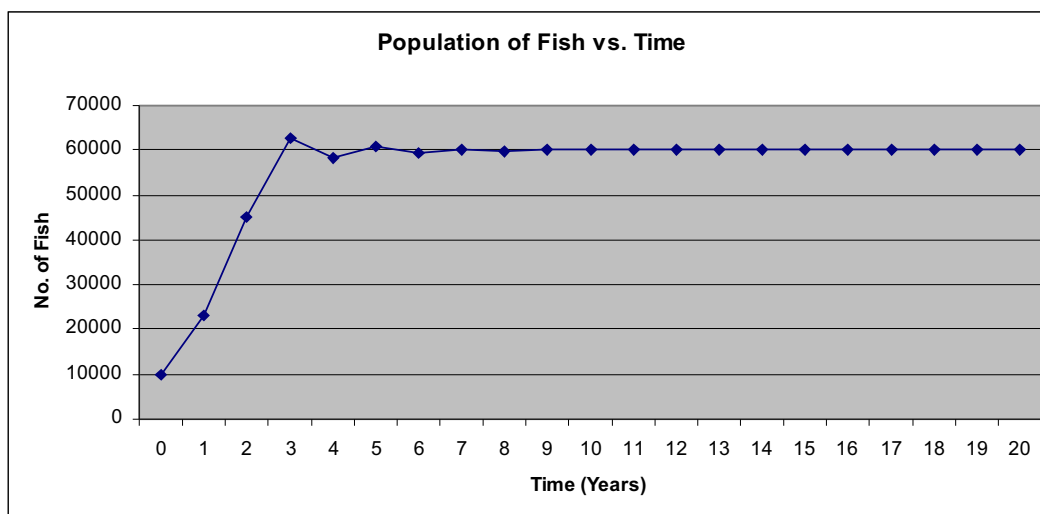| 8 | 59848.89139 | 1.003928824 |
|---|---|---|
| 9 | 60084.02714 | 0.997815294 |
| 10 | 59952.76123 | 1.001228208 |
| 11 | 60026.39569 | 0.999313712 |
| 12 | 59985.2003 | 1.000384792 |
| 13 | 60008.28214 | 0.999784664 |
| 14 | 59995.36022 | 1.000120634 |
| 15 | 60002.59772 | 0.999932459 |
| 16 | 59998.5451 | 1.000037827 |
| 17 | 60000.81469 | 0.999978818 |
| 18 | 59999.54376 | 1.000011862 |
| 19 | 60000.25549 | 0.999993357 |
| 20 | 59999.85692 | 1.00000372 |

As we have done before, we should also use the GDC to find an estimate for the logistic function:

```
LogisticReg
  a =7.39138105
  b =1.6063325
  c =60417.5108
  MSe=4.2229E+06
y=c÷(1+a·e^(-bx))
```

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However, the function that the calculator gives is just an estimate, so it is safe for us to round off this function to:

$$y = \frac{60000}{1 + 7.4e^{-1.56}}$$

Putting this data into a graph, we have:

**Population of Fish vs. Time**

(graph: No. of Fish vs Time (Years), values rising from ~10000 at year 0 to plateau near 60000)

Model for Growth Rate = 2.5

When we have a new initial growth rate, the ordered pairs i.e. $(u_n, r_n)$ have now changed to:
(60000, 1)
(10000, 2.5)

To find the linear growth factor, we form the two equations:
$2.5 = m(10000) + b$
$1 = m(60000) + b$

anX+bnY=Cn
```
        a         b        c
1[ 10000    |    2.5]
2[ 60000    |     1]

                        60000
SOLV DEL  CLR EDIT
```

anX+bnY=Cn
```
X[ -3E-5]
Y[  2.8]

                       -3E -05
REPT
```

Solving this on the GDC, we have *m* = -0.00003 and *b* = 2.8 so the linear growth factor is:

$r_n = -0.00003 \times u_n + 2.8$

and therefore the function for $u_{n+1}$ for is:

$u_{n+1} = (-0.00003 \times u_n + 2.8) \, u_n$

If we let 10000 fish with an initial growth rate of 2.5 cultivate for 20 years, the population of growth would be:

| Year | Population of Fish | Growth Rate |
|------|--------------------|-------------|
| $t$ | $u_{n+1} = (-0.00003 \times u_n + 2.8) \, u_n$ | $r_n = -0.00003 \times u_n + 2.8$ |
| 0 | 10000 | 2.5 |
| 1 | 25000 | 2.05 |
| 2 | 51250 | 1.2625 |
| 3 | 64703.125 | 0.85890625 |
| 4 | 55573.91846 | 1.132782446 |
| 5 | 62953.1593 | 0.911405221 |
| 6 | 57375.83806 | 1.078724858 |
| 7 | 61892.74277 | 0.943217717 |
| 8 | 58378.33153 | 1.048650054 |
| 9 | 61218.44052 | 0.963446784 |
| 10 | 58980.70967 | 1.03057871 |

| 11 | 60784.26368 | 0.976472089 |
| 12 | 59354.13697 | 1.019375891 |
| 13 | 60504.17625 | 0.984874712 |
| 14 | 59589.03319 | 1.012329004 |
| 15 | 60323.70664 | 0.990288801 |
| 16 | 59737.89111 | 1.007863267 |
| 17 | 60207.62608 | 0.993771218 |
| 18 | 59832.60588 | 1.005021824 |
| 19 | 60133.07467 | 0.99600776 |
| 20 | 59893.00899 | 1.00320973 |

As we have done before, we should also use the GDC to find an estimate for the logistic function:

```
LogisticReg
   a =8.02730509
   b =1.88002281
   c =60344.5353
   MSe=8.853ε+06
y=c÷(1+a·e^(-bx))
                    COPY DRAW
```

However, the function that the calculator gives is just an estimate, so it is safe for us to round off this function to:

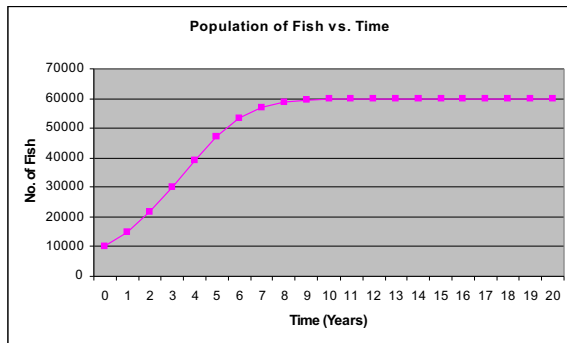$$y = \frac{60000}{1 + 8e^{-1.8x}}$$

Putting these data values into a graph, we have:
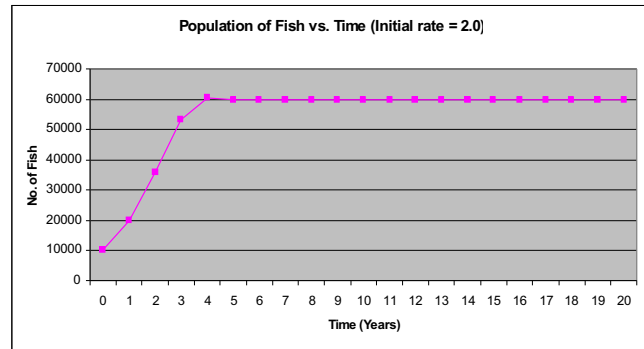
**Population of Fish vs Time**
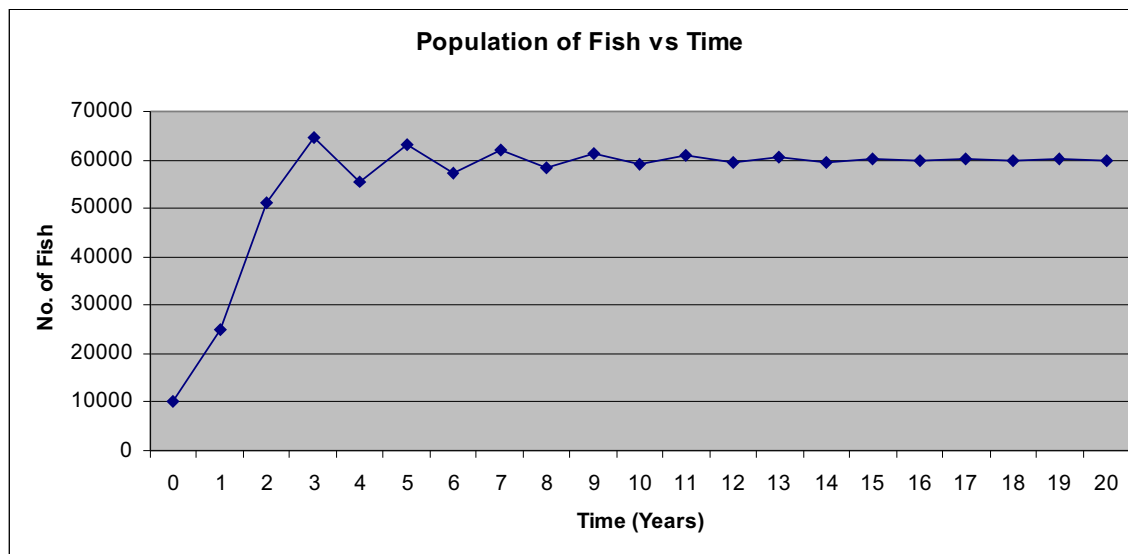
For now, let us compare these graphs to make generalizations about how the initial growth rate can effect the population of fish over time.

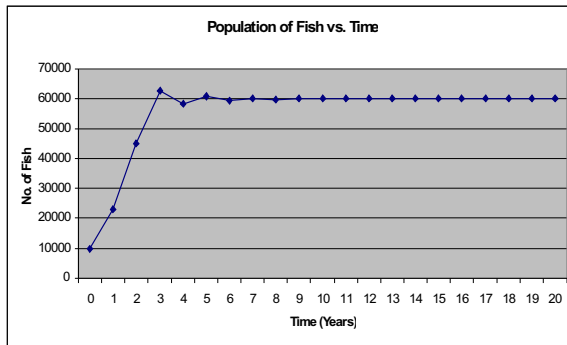The next page shows graphs the population of fish starting from initial growth rates r = 1.5, 2, 2.3 and 2.5.
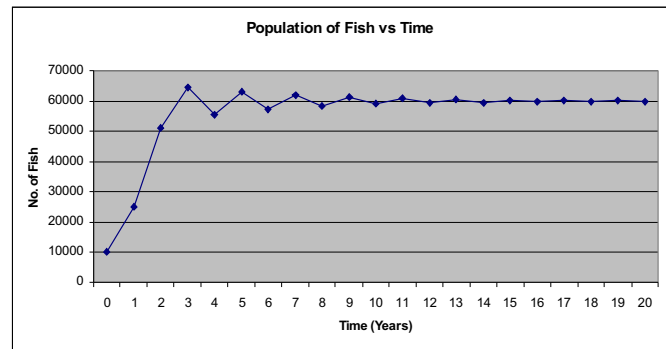
### Growth Rate = 1.5



### Growth Rate = 2



### Growth Rate = 2.3



### Growth Rate = 2.5



From these graphs, we can make a few generalizations:

Generalization 1:  As initial growth rate increases, the initial slope of the curve becomes steeper.

Generalization 2:  As initial growth rate increases, the fluctuations around the sustainable limits are larger, causing the population to settle slower.

Generalization 3:  No matter the initial rate, the sustainable limit always remains at 60000.

We should also compare the logistic function of each of these initial growth rates.

For r = 1.5, we have $y = \dfrac{60000}{1 + 6e^{-0.6x}}$

For r = 2, we have $y = \dfrac{60000}{1 + 7e^{-1.2x}}$

For r = 2.3, we have $y = \dfrac{60000}{1 + 7.4e^{-1.56}}$

For r = 2.5, we have $y = \dfrac{60000}{1 + 8e^{-1.8x}}$

First, sorting out these logistic formulas into tabular form, we can represent this as:

| Initial Growth Rate (r) | $a_{logistic}$ | $b_{logistic}$ | $c_{logistic}$ |
|---|---|---|---|
| 1.5 | 6 | 0.6 | 60000 |
| 2 | 7 | 1.2 | 60000 |
| 2.3 | 7.4 | 1.56 | 60000 |
| 2.5 | 8 | 1.8 | 60000 |

From this table, we can also make a few generalisations:

Generalistaion 1:    As *r* increases by 0.5, *a* increases by 1.

Generalisation 2:    *c* is always constant, and it is possible that this number represents the sustainable limit of the fish population.

Generalisation 3:    When I found the formula for the linear growth factor, I saw that it took the form *mx+b* where b is a constant. We can see from this data that $b_{logistic}$ = b-1.
    (i.e. for r = 1.5, linear growth factor = $-0.00001u_n + 1.6$, and $b_{logistic} = 0.6$)

It must be noted however, that these generalizations are made from only 4 examples chosen, and though they may seem to fit in to the trend, more evidence is needed to prove these generalizations.

Looking at the logistic model, we can show that in theory, $a_{logistic} = 5$, given that we know $c_{logistic} = 60000 =$ sustainable limit.

Since we know that in every case, when time $= 0$, that is, initially our population is 10000, substituting $x = 0$ in the logistic function, it must be the case that $y = 10000$. In other words:

$$10000 = \frac{60000}{1 + ae^0}$$

$$10000 = \frac{60000}{1 + a}$$

$a = 5$

Although the initial growth rate should not affect the value of $a$, and $a$ should always have a value of 5, the calculations by the calculator are not the same. So why is this not the case for the logistic functions calculated by the GDC?

The model that we used to insert data values for the population of fish is only an estimate. For example $u_{n+1} = (-0.00003 \times u_n + 2.8) u_n$ was the function that we used to estimate the values for population size. Remember that I based this function on a *linear growth factor*, and in reality, this is not the case. The raw data that I calculated on excel do not necessarily follow the same form as the geometric population growth function.

Another reason is that the calculator itself is not perfect. The logistic function calculated by the GDC is simply an estimate of the line of best fit, according to the data values we give. Since I was only able to insert approximately 20 data values, the calculator would not have had enough information to produce a reliable and accurate logistic equation.

A peculiar outcome can be observed for situations where initial growth rate is exceptionally high. To investigate this further, I will now choose to explore the fish population with initial growth rates of 2.9, 3.2 and 3.5.

<u>Model for Growth Rate = 2.9</u>

When we have a new initial growth rate, the ordered pairs i.e. ($u_n$, $r_n$) have now changed to:

(60000, 1)
(10000, 2.9)

To find the linear growth factor, we form the two equations:

$2.9 = m(10000) + b$
$1 = m(60000) + b$



Solving this on the GDC, we have *m* = -0.000038 and *b* = 2.8 so the linear growth factor is:

$r_n = -0.000038 \times u_n + 3.28$

and therefore the function for $u_{n+1}$ for is:

$u_{n+1} = (-0.000038 \times u_n + 3.28) \, u_n$

If we let 10000 fish with an initial growth rate of 2.9 cultivate for 20 years, the population of growth would be:

| Year | Population of Fish | Growth Rate |
|---|---|---|
| *t* | $u_{n+1} = (-0.000038 \times u_n + 3.28) \times u_n$ | $r_n = -0.000038 \times u_n + 3.28$ |
| 0 | 10000 | 2.9 |
| 1 | 29000 | 2.178 |
| 2 | 63162 | 0.879844 |
| 3 | 55572.70673 | 1.168237144 |
| 4 | 64922.10021 | 0.812960192 |
| 5 | 52779.08305 | 1.274394844 |
| 6 | 67261.39131 | 0.72406713 |
| 7 | 48701.76257 | 1.429333022 |
| 8 | 69611.03749 | 0.634780576 |

| 9 | 44187.73444 | 1.600866091 |
| 10 | 70738.64572 | 0.591931463 |
| 11 | 41872.43004 | 1.688847659 |
| 12 | 70716.15543 | 0.592786094 |
| 13 | 41919.55354 | 1.687056965 |
| 14 | 70720.67479 | 0.592614358 |
| 15 | 41910.08729 | 1.687416683 |
| 16 | 70719.78048 | 0.592648342 |
| 17 | 41911.96064 | 1.687345496 |
| 18 | 70719.958 | 0.592641596 |
| 19 | 41911.58878 | 1.687359626 |
| 20 | 70719.92278 | 0.592642934 |

An estimate for the logistic function would be :

$$y = \frac{57984.6612}{(1 + 11.6232775)e^{-2.67618108\,x}}$$

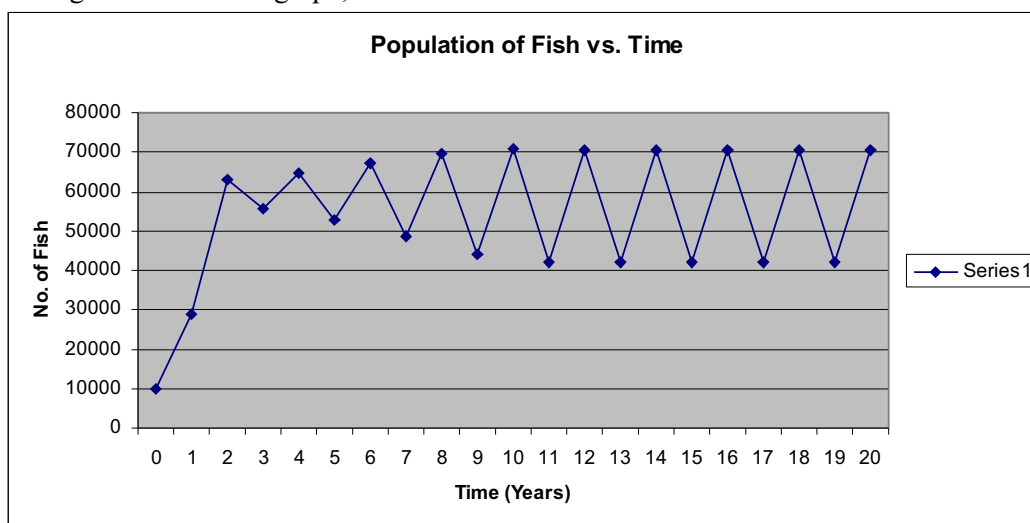When we round this off, it becomes approximately:

$$y = \frac{60000}{(1 + 11)e^{-2.28\,x}}$$

Although the rounding off errors were relatively large this time, (i.e. from 57984.6612 to become 60000, or from 2.67618108 to become 2.28), this does not mean that I am *incorrect*. First let us take a look at the graph of the data shown in the table above.

Putting this data into a graph, we have:

**Population of Fish vs. Time**

*No. of Fish* (y-axis: 0 to 80000)
*Time (Years)* (x-axis: 0 to 20)

Series 1

The first thing that we can see from this graph is that it never reaches a stable limit. The population of fish simply oscillates and fluctuates around the sustainable limit (60000). Unlike in a lake where initial growth rate is relatively small (i.e. r = 1.5, 2, 2.3, 2.5) which slowly converges to a sustainable limit at 60000, with a high *r*, this does not happen. This will greatly affect the estimate of the logistic function by the GDC.

This is because the GDC estimates the logistic function by finding a line of best fit through the data we insert, and finds an equation for the line of best fit. When such large fluctuations exist in a graph (which I have only provided 20 values for), the error in the estimate will also be much greater. Though my rounding off of the formula may have been great, and have done so in order to support my previous generalizations on logistic formulas, I am in a way, justified to do so.

Having said, this, this logistic function model has sufficiently proven my generalization 1 for logistic models wrong. I had stated that "As *r* increases by 0.5, *a* increases by 1." This is clearly not the case, and so cannot be true.
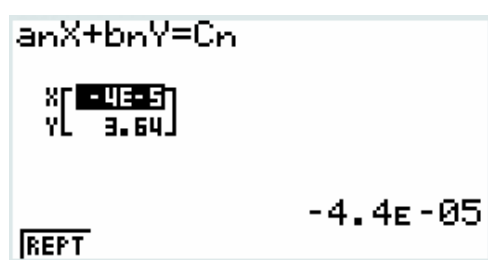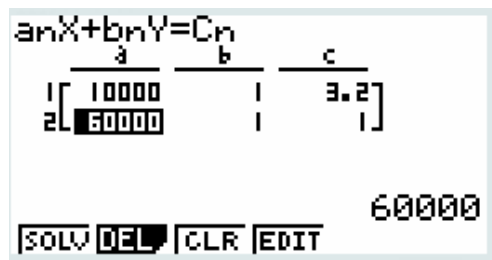
Model for Growth Rate = 3.2

When we have a new initial growth rate, the ordered pairs i.e. ($u_n$, $r_n$) have now changed to:

(60000, 1)
(10000, 3.2)

To find the linear growth factor, we form the two equations:

$3.2 = m(10000) + b$
$1 = m(60000) + b$



Solving this on the GDC, we have *m* = -0.000044 and *b* = 3.64 so the linear growth factor is:

$r_n = -0.000044 \times u_n + 3.64$

and therefore the function for $u_{n+1}$ for is:

$u_{n+1} = (-0.000044 \times u_n + 3.64) \, u_n$

If we let 10000 fish with an initial growth rate of 3.2 cultivate for 20 years, the population of growth would be:

| Year | Population of Fish | Growth Rate |
|------|--------------------|-------------|
| $t$ | $u_{n+1} = (-0.000044 \times u_n + 3.64) \times u_n$ | $r_n = -0.000044 \times u_n + 3.64$ |
| 0 | 10000 | 2.9 |
| 1 | 32000 | 2.232 |
| 2 | 71424 | 0.497344 |
| 3 | 35522.29786 | 2.077018894 |
| 4 | 73780.48382 | 0.393658712 |
| 5 | 29044.33023 | 2.36204947 |
| 6 | 68604.14483 | 0.621417628 |
| 7 | 42631.82492 | 1.764199703 |
| 8 | 75211.05288 | 0.330713673 |
| 9 | 24873.32356 | 2.545573763 |
| 10 | 63316.87986 | 0.854057286 |
| 11 | 54076.24259 | 1.260645326 |
| 12 | 68170.96248 | 0.640477651 |
| 13 | 43661.97792 | 1.718872972 |
| 14 | 75049.39373 | 0.337826676 |
| 15 | 25353.68721 | 2.524437763 |
| 16 | 64003.80542 | 0.823832562 |
| 17 | 52728.41897 | 1.319949565 |
| 18 | 69598.8537 | 0.577650437 |
| 19 | 40203.80828 | 1.871032436 |
| 20 | 75222.62933 | 0.33020431 |

Although normally, I would look for a logistic function, however, seeing the fluctuation of results after seeing the graph (see next page), I have concluded that the fluctuations and oscillations of population of fish around the sustainable limit is far too great for the estimated logistic function to be precise, and the function obtained would have so much error that no conjectures or generalisations can be formed from it.

Following the same method as above, the graph for an initial growth rate of 3.2 is:
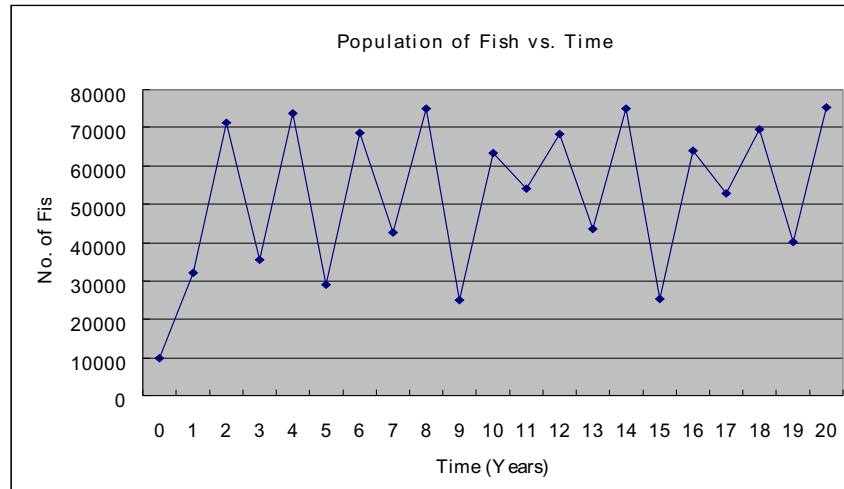


During smaller initial growth rates (i.e. $r = 1.5, 2, 2.3, 2.5$), it can be said that over time, the population size of fish will settle down and converge at a population of 60000.

When we look at higher initial growth rates however, (i.e. $r = 2.9, 3.2$), the population of fish does not converge towards the sustainable limit, but fluctuates around it. Looking at the graph for r=2.9, we see that these fluctuations in fact, get larger, until it settles down at a certain limit. The same can be said for the graph for r=3.2, where the population of fish oscillates in a pattern over time.

Model for Growth Rate = 3.5

When we have a new initial growth rate, the ordered pairs i.e. ($u_n$, $r_n$) have now changed yet again to:

(60000, 1)
(10000, 3.5)

To find the linear growth factor, we form the two equations:

$3.5 = m(10000) + b$
$1 = m(60000) + b$



Using the GDC to solve this, we have that $m = -0.00006$ and $b = 4.6$ so the linear growth factor is:
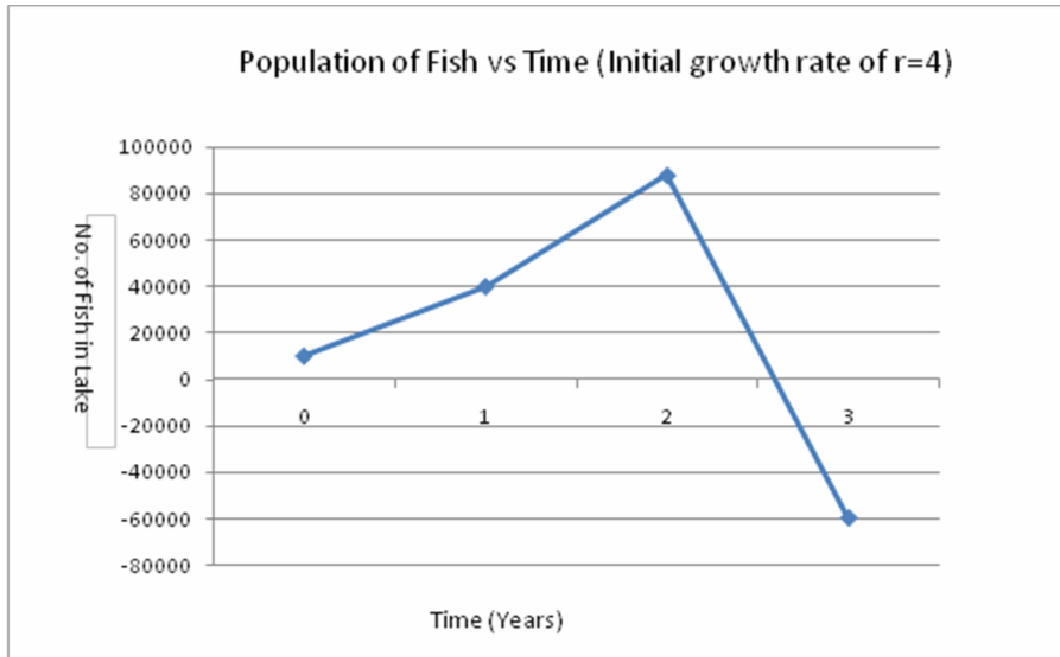
$r_n = -0.00006 \times u_n + 4.6$

and therefore the function for $u_{n+1}$ for is:

$u_{n+1} = (-0.00006 \times u_n + 4.6) \, u_n$

If we let 10000 fish with an initial growth rate of 3.5 cultivate for a while, the population of growth would be:

| Year | Population of Fish | Growth Rate |
|------|--------------------|-------------|
| $t$ | $u_{n+1} = (-0.00006 \times u_n + 4.6) \times u_n$ | $r_n = -0.00006 \times u_n + 4.6$ |
| 0 | 10000 | 2.9 |
| 1 | 40000 | 2.2 |
| 2 | 88000 | -0.68 |
| 3 | -59840 | N/A |

Putting this in a graph, we have:



We see that by the third year, the population of fish would have exterminated to 0 (a negative number of fish is impossible of course). We can conclude from this that as the initial growth rate of fish increases, the fluctuations and oscillations of population of fish every year will become greater, so much to the point that the fluctuations may lead to a population of fish 'below zero'.

Although we only have data for the three years that the fish existed, the graph shows hints of trying to take the 'S-shape' form as the other graphs did (i.e. r = 2.9, 3.2), however its growth rate was much too high in the beginning, causing the population of fish to soar much above the sustainable limit. Since when there are 60000 fish, growth rate is 1, any population of fish higher than this will result in a growth rate of less than 1 i.e. number of fish will diminish. In this case, the number of fish has diminished to such an extreme extent that there are no fish left.

So far we have been looking at examples where a population of fish has been left alone to cultivate. In real life situations however, it is most likely that governments or other organizations would like to harvest from this population of fish. Therefore, another area that is worthy of our investigation is how a harvest would affect the population of fish in the lake.

Let us assume that the population of fish in a lake follows the first model made in this portfolio. That is:

$r_{10000} = 1.5$, $r_{60000} = 1$

this also means that the population of fish follows the equations that I had calculated before:

$r_n = -0.00001 \times u_n + 1.6$
and
$u_{n+1} = (-0.00001 \times u_n + 1.6) \times u_n$

Let us say that a government body wishes to harvest from a population of fish, AFTER its population has settled (i.e. reached its sustainable limit). This means that until there is 60000 fish in the lake, no harvesting will take place, so 60000 is our 'initial number of fish'.

Harvest Size of 5000

Imagine that 5000 fish were taken from the lake at the end of each year. How would that affect the population of fish in the lake? It is best to first draw up a table of values:

| Year | No. of Fish at beginning of year | Rate |
|------|----------------------------------|------|
| $t$ | $u_{n+1} = u_n \times r_n - 5000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 55000 | 1.05 |
| 2 | 52750 | 1.0725 |
| 3 | 51574.375 | 1.08425625 |
| 4 | 50919.83843 | 1.090801616 |
| 5 | 50543.44203 | 1.09456558 |
| 6 | 50323.11193 | 1.096768881 |
| 7 | 50192.82314 | 1.098071769 |
| 8 | 50115.32208 | 1.098846779 |
| 9 | 50069.06026 | 1.099309397 |
| 10 | 50041.38846 | 1.099586115 |
| 11 | 50024.81595 | 1.099751841 |
| 12 | 50014.88341 | 1.099851166 |
| 13 | 50008.92783 | 1.099910722 |
| 14 | 50005.3559 | 1.099946441 |
| 15 | 50003.21325 | 1.099967867 |
| 16 | 50001.92785 | 1.099980722 |
| 17 | 50001.15667 | 1.099988433 |
| 18 | 50000.69399 | 1.09999306 |
| 19 | 50000.41639 | 1.099995836 |
| 20 | 50000.24983 | 1.099997502 |

Putting the data for the population of fish in the last 20 years (following this model) in a graph, we have:
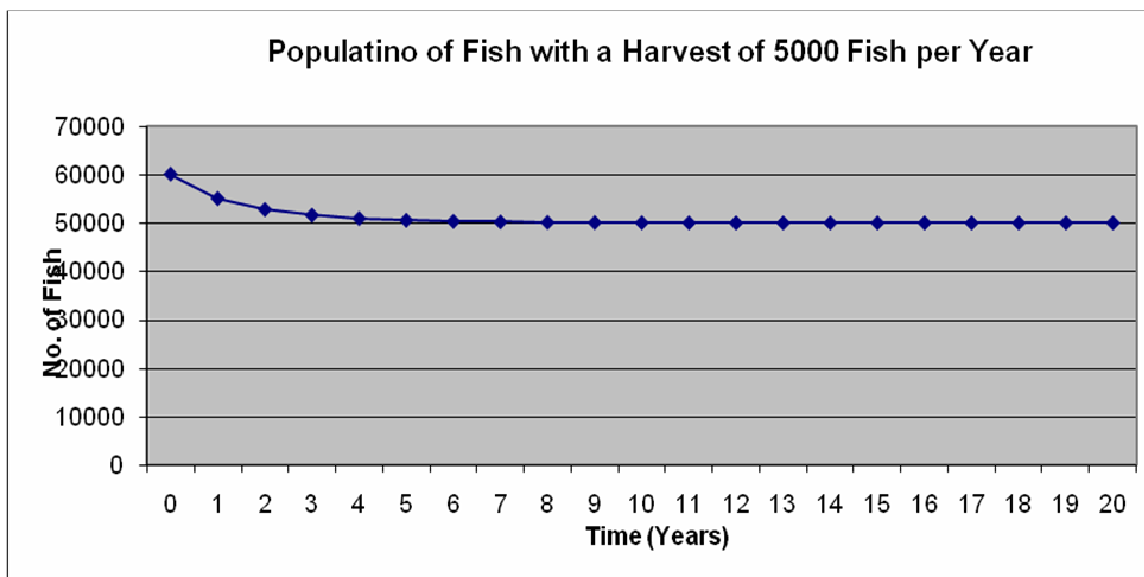


As we can see from this graph, the long term sustainable limit of fish has dropped from 60000 to approximately 50000. So, in a way, we can say that it is feasible for people to harvest 5000 fish per annum, as a small harvest such as this would not deplete the population of fish, even though it will lower the maximum sustainable limit of fish.

Since Rate = *-0.00001 × $u_n$ +1.6*, as the population of fish decrease, the rate of growth increases so that at some no. of fish, there will be an increase in 5000 fish due to growth, and a decrease in 5000 fish due to harvest. This leads to a new stable sustainable limit of fish.

Next, we should explore other harvest sizes to see how they effect the sustainable limit of the fish population. I will now explore harvest sizes 3000, 4000, 5000 (already done), 6000, 7000, 8000 and 9000.

The general formula for finding population of fish when there is a harvest is:

$u_n = [(-0.00001 × u_n + 1.6) × u_n] – H$

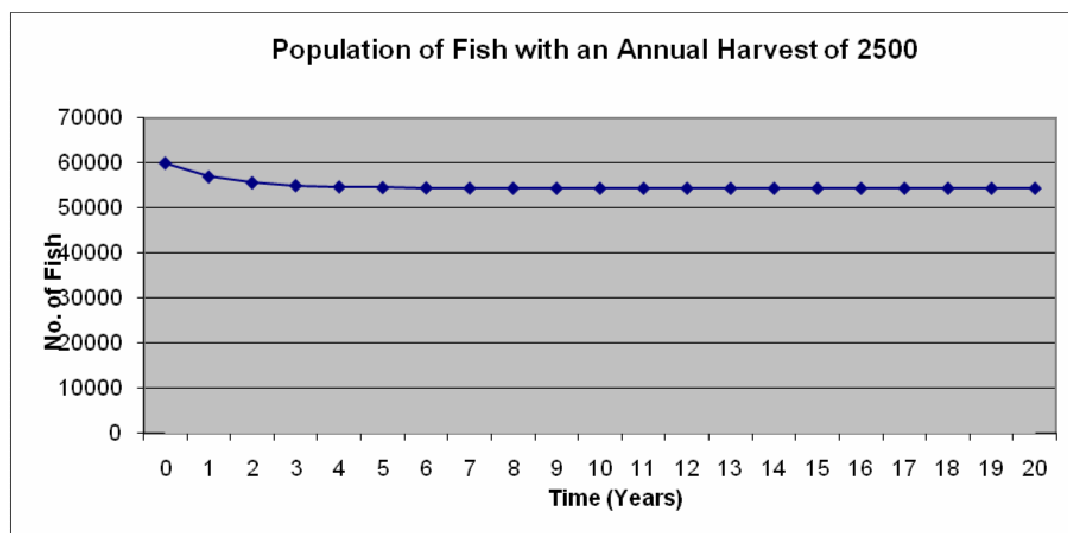where H is the harvest size per year.

Harvest Size of 3000

We first use Excel to tabulate the data for us:

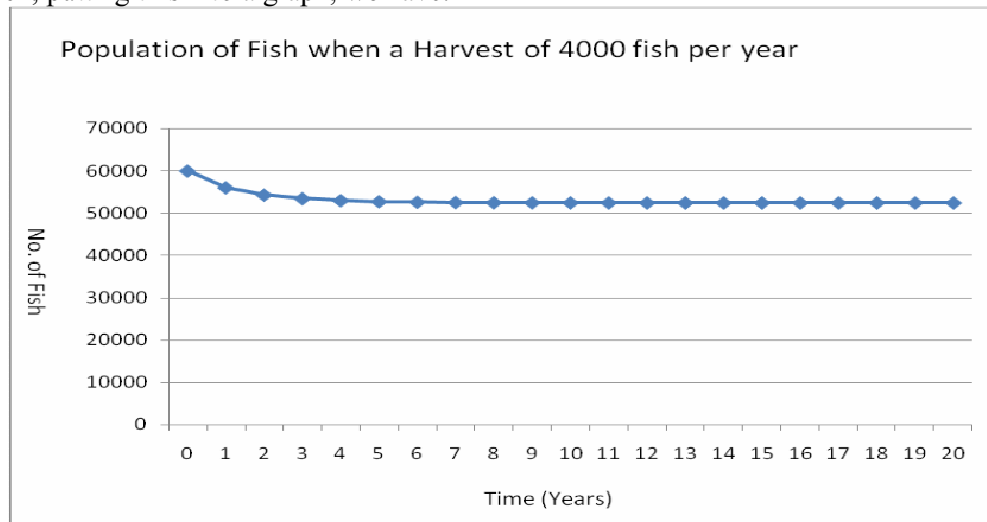| Year | No. of Fish at beginning of year | Growth Rate |
|---|---|---|
| $t$ | $u_{n+1} = u_n \times r_n - 3000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 57000 | 1.03 |
| 2 | 55710 | 1.0429 |
| 3 | 55099.959 | 1.04900041 |
| 4 | 54799.87958 | 1.052001204 |
| 5 | 54649.53931 | 1.053504607 |
| 6 | 54573.54143 | 1.054264586 |
| 7 | 54534.95204 | 1.05465048 |
| 8 | 54515.31333 | 1.054846867 |
| 9 | 54505.30745 | 1.054946925 |
| 10 | 54500.20652 | 1.054997935 |
| 11 | 54497.60532 | 1.055023947 |
| 12 | 54496.27866 | 1.055037213 |
| 13 | 54495.60198 | 1.05504398 |
| 14 | 54495.25681 | 1.055047432 |
| 15 | 54495.08075 | 1.055049192 |
| 16 | 54494.99094 | 1.055050091 |
| 17 | 54494.94513 | 1.055050549 |
| 18 | 54494.92176 | 1.055050782 |
| 19 | 54494.90984 | 1.055050902 |
| 20 | 54494.90376 | 1.055050962 |

Then, putting this into a graph, we have:



The sustainable limit seems to have dropped to approximately 54494 fish.

Harvest Size of 4000

We first use Excel to tabulate the data for us:

| Year | No. of Fish at beginning of year | Growth Rate |
|---|---|---|
| $t$ | $u_{n+1} = u_n \times r_n - 4000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 56000 | 1.04 |
| 2 | 54240 | 1.0576 |
| 3 | 53364.224 | 1.06635776 |
| 4 | 52905.35437 | 1.070946456 |
| 5 | 52658.80178 | 1.073411982 |
| 6 | 52524.5888 | 1.074754112 |
| 7 | 52451.01779 | 1.075489822 |
| 8 | 52410.53579 | 1.075894642 |
| 9 | 52388.21465 | 1.076117854 |
| 10 | 52375.8931 | 1.076241069 |
| 11 | 52369.08718 | 1.076309128 |
| 12 | 52365.32657 | 1.076346734 |
| 13 | 52363.24824 | 1.076367518 |
| 14 | 52362.09952 | 1.076379005 |
| 15 | 52361.46457 | 1.076385354 |
| 16 | 52361.11359 | 1.076388864 |
| 17 | 52360.91958 | 1.076390804 |
| 18 | 52360.81234 | 1.076391877 |
| 19 | 52360.75305 | 1.076392469 |
| 20 | 52360.72028 | 1.076392797 |

Then, putting this into a graph, we have:



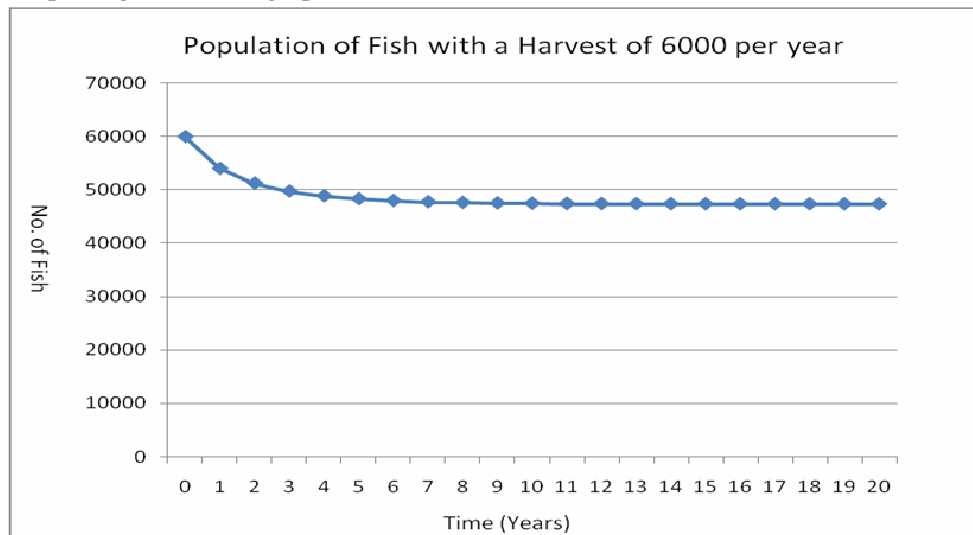The sustainable limit seems to have dropped to approximately 52360 fish.

Harvest Size of 6000

We first use Excel to tabulate the data for us:

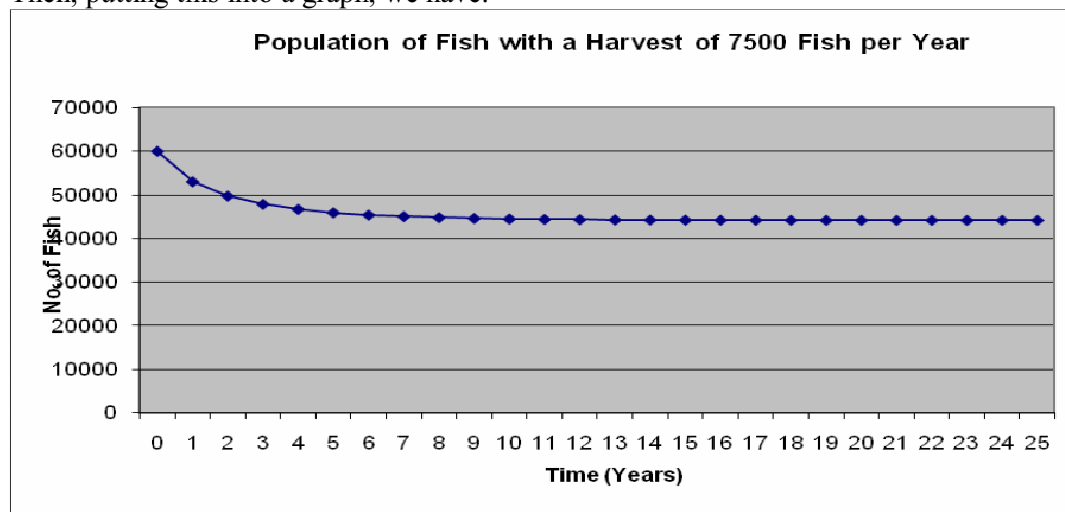| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 6000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 54000 | 1.06 |
| 2 | 51240 | 1.0876 |
| 3 | 49728.624 | 1.102714 |
| 4 | 48836.43795 | 1.111636 |
| 5 | 48288.324 | 1.117117 |
| 6 | 47943.69606 | 1.120563 |
| 7 | 47723.93377 | 1.122761 |
| 8 | 47582.55549 | 1.124174 |
| 9 | 47491.09291 | 1.125089 |
| 10 | 47431.7096 | 1.125683 |
| 11 | 47393.0646 | 1.126069 |
| 12 | 47367.87764 | 1.126321 |
| 13 | 47351.4459 | 1.126486 |
| 14 | 47340.71915 | 1.126593 |
| 15 | 47333.71375 | 1.126663 |
| 16 | 47329.13742 | 1.126709 |
| 17 | 47326.14739 | 1.126739 |
| 18 | 47324.19355 | 1.126758 |
| 19 | 47322.91673 | 1.126771 |
| 20 | 47322.08229 | 1.126779 |

Then, putting this into a graph, we have:



This time, the sustainable limit of fish has dropped to approximately 47322 fish.

Harvest Size of 7000

We first use Excel to tabulate the data for us:

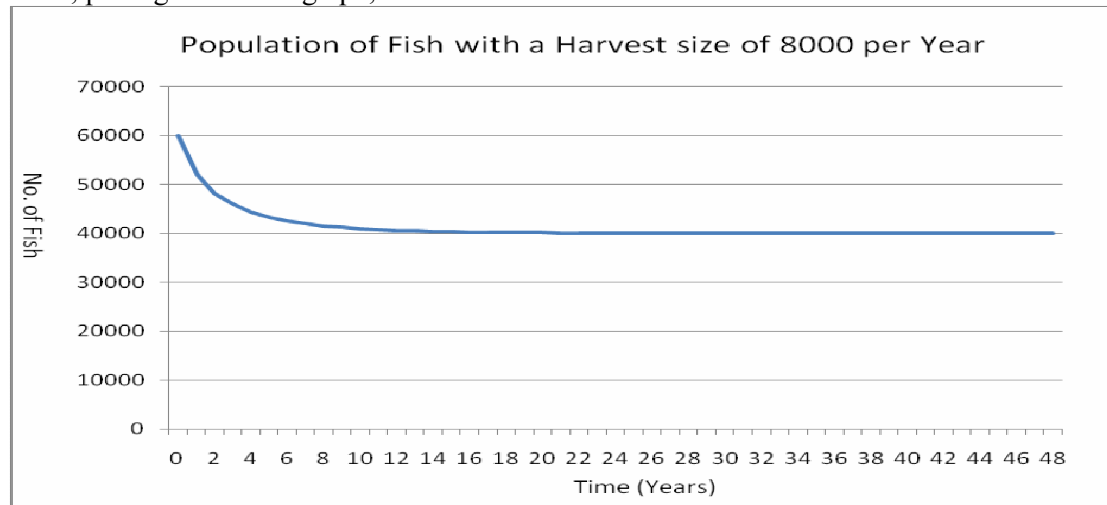| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 7000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 53000 | 1.07 |
| 2 | 49710 | 1.1029 |
| 3 | 47825.159 | 1.12175 |
| 4 | 46647.79607 | 1.13352 |
| 5 | 45876.30493 | 1.14124 |
| 6 | 45355.73435 | 1.14644 |
| 7 | 44997.74857 | 1.15002 |
| 8 | 44748.42395 | 1.15252 |
| 9 | 44573.26386 | 1.15427 |
| 10 | 44449.46366 | 1.15551 |
| ......... | ....................................... | ................... |
| 25 | 44144.20251 | 1.15856 |
| 26 | 44143.61786 | 1.15856 |
| 27 | 44143.1986 | 1.15857 |
| 28 | 44142.89793 | 1.15857 |
| 29 | 44142.68231 | 1.15857 |
| 30 | 44142.52768 | 1.15857 |
| 31 | 44142.41679 | 1.15858 |
| 32 | 44142.33726 | 1.15858 |
| 33 | 44142.28023 | 1.15858 |
| 34 | 44142.23933 | 1.15858 |

Then, putting this into a graph, we have:



The sustainable limit of fish has become somewhere close to 44142 fish.

Harvest Size of 8000

We first use Excel to tabulate the data for us:

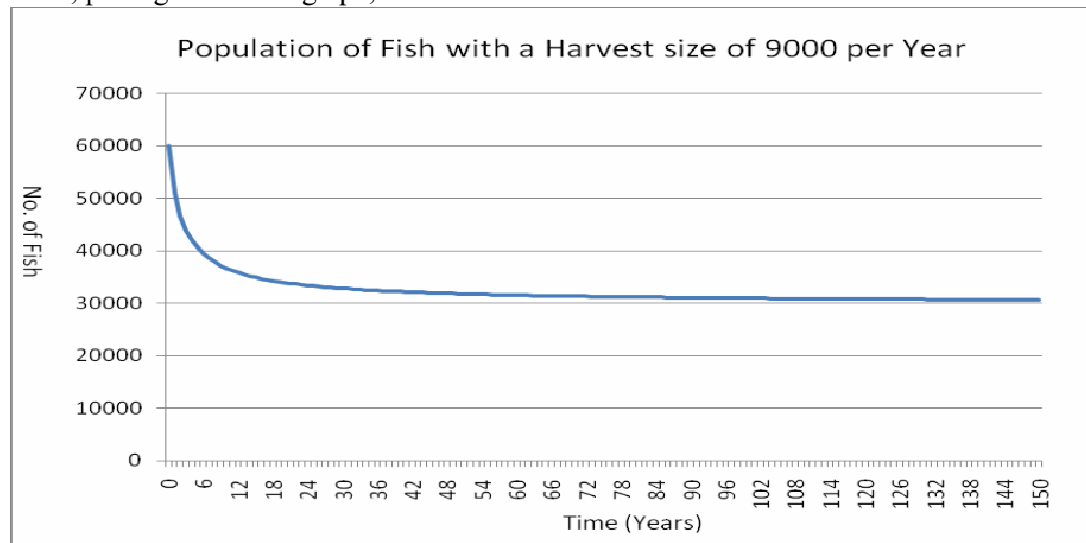| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 8000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 52000 | 1.08 |
| 2 | 48160 | 1.1184 |
| 3 | 45862.144 | 1.141379 |
| 4 | 44346.06788 | 1.156539 |
| 5 | 43287.97124 | 1.16712 |
| 6 | 42522.26944 | 1.174777 |
| 7 | 41954.19712 | 1.180458 |
| 8 | 41525.16884 | 1.184748 |
| 9 | 41196.87367 | 1.188031 |
| 10 | 40943.17387 | 1.190568 |
| ……… | …………………………. | …………….. |
| 35 | 40003.36435 | 1.199966 |
| 36 | 40002.69136 | 1.199973 |
| 37 | 40002.15302 | 1.199978 |
| 38 | 40001.72237 | 1.199983 |
| 39 | 40001.37787 | 1.199986 |
| 40 | 40001.10227 | 1.199989 |
| 41 | 40000.88181 | 1.199991 |
| 42 | 40000.70544 | 1.199993 |
| 43 | 40000.56434 | 1.199994 |
| 44 | 40000.45147 | 1.199995 |

Then, putting this into a graph, we have:



The sustainable limit for a harvest size of 8000 is roughly 40000 fish.

Harvest Size of 9000

We first use Excel to tabulate the data for us:

| Year | No. of Fish at beginning of year | Growth Rate |
|---|---|---|
| $t$ | $u_{n+1} = u_n \times r_n - 9000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 51000 | 1.09 |
| 2 | 46590 | 1.1341 |
| 3 | 43837.719 | 1.16162281 |
| 4 | 41922.89433 | 1.180771057 |
| 5 | 40501.34024 | 1.194986598 |
| 6 | 39398.55877 | 1.206014412 |
| 7 | 38515.2297 | 1.214847703 |
| 8 | 37790.13833 | 1.222098617 |
| 9 | 37183.27578 | 1.228167242 |
| 10 | 36667.28127 | 1.233327187 |
| ……… | …………………… | ……………… |
| 7991 | 30012.49639 | 1.299875036 |
| 7992 | 30012.49483 | 1.299875052 |
| 7993 | 30012.49326 | 1.299875067 |
| 7994 | 30012.4917 | 1.299875083 |
| 7995 | 30012.49014 | 1.299875099 |
| 7996 | 30012.48858 | 1.299875114 |
| 7997 | 30012.48702 | 1.29987513 |
| 7998 | 30012.48546 | 1.299875145 |
| 7999 | 30012.48391 | 1.299875161 |
| 8000 | 30012.48235 | 1.299875177 |

Then, putting this into a graph, we have:



The new sustainable limit lies at approximately 30012.

From these tables and graphs, we realize that for every harvest size, there is a new sustainable limit, and in fact we can make generalizations about these as well, first by drawing a table as follows:

| Harvesting Size | Approximate Sustainable Limit |
|---|---|
| 0 | 60000 |
| 1000 | ? |
| 2000 | ? |
| 3000 | 54494 |
| 4000 | 52360 |
| 5000 | 50000 |
| 6000 | 47322 |
| 7000 | 44142 |
| 8000 | 40000 |
| 9000 | 30000 |

It is obvious from this that as the harvesting size increases, the sustainable limit decreases. This is feasible will our common sense as well, since as more fish are taken away, growth rate needs to be higher to make up for the harvest; the smaller the population of fish, the greater the growth rate.

Anyhow, putting these values in a graph, we have:
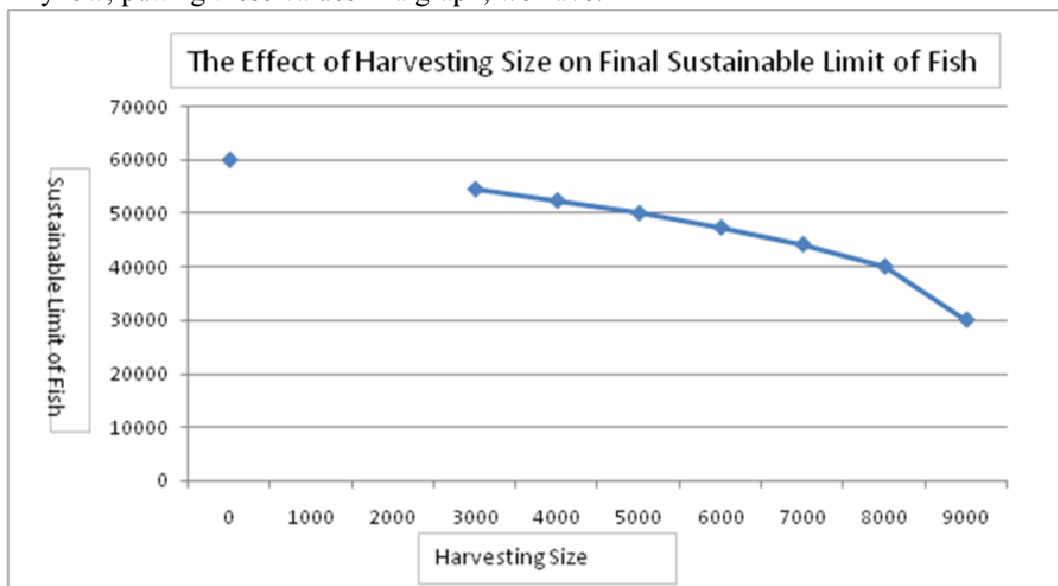


From the graph, however, we can also see that the relationship between the sustainable limit of fish and annual harvest size is NOT directly proportional, but instead seems to be a quadratic, as the line is shaped like an inverse parabola.

With this new information I have achieved, I can now continue to explore the effects of harvests of fish, in realistic conditions. If a government body wished to harvest from the lake of fish, it is most important to find the *maximum* harvest of fish which would allow for the population of fish to remain i.e. the fish population will not deplete.

First of all, I decided to use a 'trial and error' method of approaching this question. Since earlier I had shown that a harvest of 9000 would NOT deplete the fish population, the investigation of a harvest size of 10000 was next.

<u>Harvest Size of 10000</u>

As before, I used Excel to tabulate the data for us:

| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 10000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 50000 | 1.1 |
| 2 | 45000 | 1.15 |
| 3 | 41750 | 1.1825 |
| 4 | 39369.375 | 1.20630625 |
| 5 | 37491.52312 | 1.225084769 |
| 6 | 35930.29393 | 1.240697061 |
| 7 | 34578.61007 | 1.254213899 |
| 8 | 33368.97337 | 1.266310266 |
| 9 | 32255.47356 | 1.277445264 |
| 10 | 31204.60195 | 1.287953981 |
| 11 | 30190.09129 | 1.298099087 |
| 12 | 29189.72994 | 1.308102701 |
| 13 | 28183.16456 | 1.318168354 |
| 14 | 27150.15565 | 1.328498443 |
| 15 | 26068.93953 | 1.339310605 |
| 16 | 24914.40716 | 1.350855928 |
| 17 | 23655.77462 | 1.363442254 |
| 18 | 22253.28266 | 1.377467173 |
| 19 | 20653.16636 | 1.393468336 |
| 20 | 18779.53337 | 1.412204666 |
| 21 | 16520.54466 | 1.434794553 |
| 22 | 13703.5875 | 1.462964125 |
| 23 | 10047.85689 | 1.499521431 |
| 24 | 5066.97675 | 1.549330233 |
| 25 | -2149.579734 | 1.621495797 |

In the 25th year, the population of fish reached a value below zero! In reality, this is impossible because you cannot have a 'negative number of fish'. In other words, the number of fish left in the lake is zero.

When we put this into a graph, we can represent this more clearly:

**Population of Fish with Harvest 10000 Fish per Year**
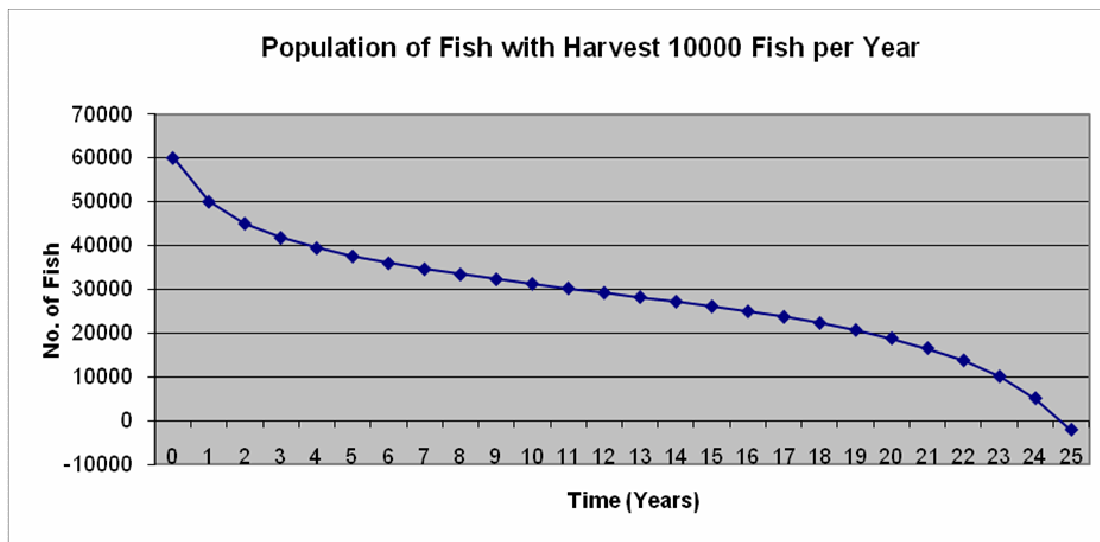


The initial downfall of population due to harvest led to a rather sharp decline in population size. However as population of fish decreased, rate of growth became larger, hence the slope of the curve 'flattening out'.

After a few years however, (namely at year 13), the decrease in population size began to speed up again. Even though the growth rate may be large, the growth rate is a percentage of the existing population.

In this case, the population has become so small that even a large growth rate cannot overcome the 10000 annual fish harvest, and so the population of fish will die out.

It is important to realize that a 'trial and error' approach is not the most effective way to method to find the 'maximum annual harvest size' for fish in the lake, as you would have to test every single harvest size.

The next thing to be done was to consider the function model for the population size of the fish in the lake. When we look closely at the equation we gave for $u_{n+1}$, we find that it is in fact, a quadratic equation, see below:

I had stated earlier that the general equation to finding the population of fish in a lake while there is an annual harvest is:

$u_{n+1} = [(-0.00001 \times u_n + 1.6) \times u_n] - H$

where $u_{n+1}$ is the total population of fish in the next year
where H is the size of annual harvest

The 'Sustainable Limit' of a population means that given more time, the population will remain constant. In other words, growth rate = 1.

Since the formula for any geometric population growth models will always take the form of $u_{n+1} = r \times u_n$, and we also have that growth rate is equal to 1 (r=1) when a population has reached a sustainable limit, we can say that $u_{n+1} = u_n$.

Now, equation for population of fish can become:

$u_n = [(-0.00001 \times u_n + 1.6) \times u_n] - H$. Expanding this, we have:

$u_n = -0.00001u_n^2 + 1.6u_n - H$

Now, we must transform this into something we can work with. To make this a quadratic equation, we must make it take the form $ax^2 + bx + c = 0$. So our equation becomes:

$-0.00001u_n^2 + 0.6u_n - H = 0$

Since the equation we have is a quadratic, it is very important to consider its discriminant. Recalling our theory on quadratics, we know that if the discriminant is:

$\Delta > 0$, quadratic has 2 solutions

$\Delta = 0$, quadratic has 1 solution

$\Delta < 0$, quadratic has 0 solutions.

Anyhow, $\Delta = b^2 - 4ac$ if the quadratic takes the form $ax^2 + bx + c = 0$. So, in our situation, the discriminant is equal to:

$\Delta = 0.6^2 - 4(-0.00001)(-H)$
$\Delta = 0.6^2 - 0.00004H$

To find the maximum annual harvest size, our aim is to maximise the value of H. In saying this, we must also ensure that our quadratic equation has at least 1 solution.

By speculation, we can see that as H is bigger, $\Delta$ becomes smaller ($b^2$- 4aH) – in other words, we are looking for the value of H where $\Delta$ is as small as possible, but still has solutions; $\Delta$ must equal 0.

Solving this equation:

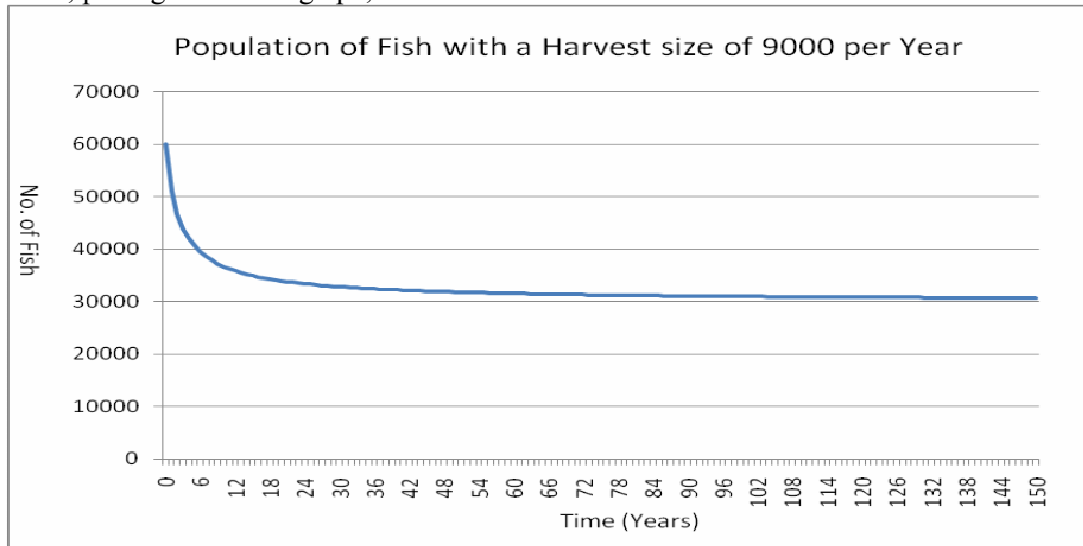$0.6^2 - 0.00004H$ $=$ $0$

$0.6^2$ $=$ $0.00004H$

$9000$ $=$ $H$

According to quadratic theory, the maximum annual harvest of fish from the lake is 9000 fish per year. To prove this, we should once again turn to Excel to give us some data on the population sizes over time.

This table has already been done previously, however it is useful to place the table in this section as well, as it is of vital importance to finding the maximum harvest size

| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 9000$ | $-0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 51000 | 1.09 |
| 2 | 46590 | 1.1341 |
| 3 | 43837.719 | 1.16162281 |
| 4 | 41922.89433 | 1.180771057 |
| 5 | 40501.34024 | 1.194986598 |
| 6 | 39398.55877 | 1.206014412 |
| 7 | 38515.2297 | 1.214847703 |
| 8 | 37790.13833 | 1.222098617 |
| 9 | 37183.27578 | 1.228167242 |
| 10 | 36667.28127 | 1.233327187 |
| ……… | …………………… | ……………… |
| 7991 | 30012.49639 | 1.299875036 |
| 7992 | 30012.49483 | 1.299875052 |
| 7993 | 30012.49326 | 1.299875067 |
| 7994 | 30012.4917 | 1.299875083 |
| 7995 | 30012.49014 | 1.299875099 |
| 7996 | 30012.48858 | 1.299875114 |

| 7997 | 30012.48702 | 1.29987513 |
| 7998 | 30012.48546 | 1.299875145 |
| 7999 | 30012.48391 | 1.299875161 |
| 8000 | 30012.48235 | 1.299875177 |

Then, putting this into a graph, we have:



This graph shows that it IS in fact feasible to harvest at 9000 fish a year, as the population of fish will NOT deplete.

Next, I will show that any harvest size greater than 9000 will lead to a depletion of the fish population.

Harvest Size of 9001

| Year | No. of Fish at beginning of year | Rate |
|------|----------------------------------|------|
| $t$ | $u_{n+1} = u_n \times r_n - 9001$ | $-0.00001 \times u_n + 1.6$ |
| 0 | 60000 | 1 |
| 1 | 50999 | 1.09001 |
| 2 | 46588.42 | 1.134116 |
| 3 | 43835.66 | 1.161643 |
| 4 | 41920.41 | 1.180796 |
| 5 | 40498.45 | 1.195016 |
| 6 | 39395.27 | 1.206047 |
| 7 | 38511.56 | 1.214884 |
| 8 | 37786.09 | 1.222139 |
| 9 | 37178.86 | 1.228211 |
| 10 | 36662.5 | 1.233375 |
| ……… | ………………….. | …………. |
| 980 | 18765.02 | 1.41235 |
| 981 | 17501.77 | 1.424982 |

| 982 | 15938.71 | 1.440613 |
| 983 | 13960.51 | 1.460395 |
| 984 | 11386.86 | 1.486131 |
| 985 | 7921.366 | 1.520786 |
| 986 | 3045.705 | 1.569543 |
| 987 | -4220.64 | 1.642206 |

We can also represent this in a graph, as there is too much data for all of it to fit on a table.
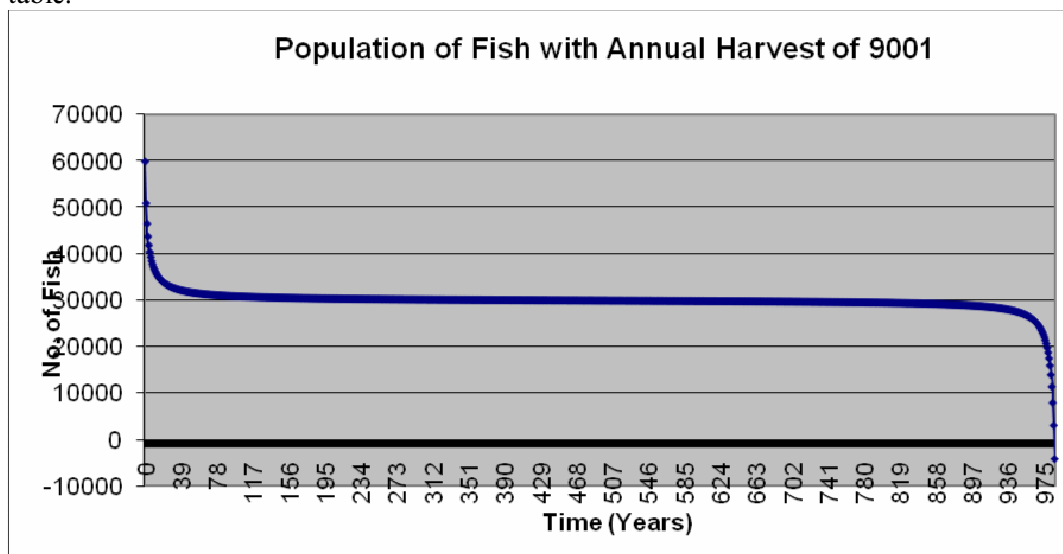


With a harvest size of 9001, we see that the fish population will deplete over time, and although slowly, after a period of 987 years, the population of fish WILL be 0. As 'number of fish' is a discrete value (i.e. it cannot have any decimal places), this proves that 9000 is the maximum harvest size.

Now, as a final check, I will solve the quadratic formula for $u_n$, by substituting the harvest size H = 9000. This is because even though the theory may show that this answer is correct, our population size $u_n$ must be between 0 and 60000. This is because our population cannot be negative, nor can it exceed the sustainable limit.

$$-0.00001u_n^2 + 0.6u_n - 9000 = 0$$

Solving this on the GDC, we have:

Since we have that $u_n$ = 30000 (there is only 1 solution because $\Delta$ = 0), which is in between 0 and 60000, the conjecture that 9000 is the maximum harvest size in this particular lake, is sufficiently proven.

Taking a closer look, we can also see that the value for $u_n$ as solved previously on our GDC ($u_n$ = 30000) is also equal to our new sustainable limit when a 9000 annual harvest is in place.

In realistic situations, government bodies may not be patient enough to wait for the population of fish to settle before harvesting. For examples where I had allowed the population of fish to settle, I used 60000 as my initial population size. In the next few examples, I will be changing the initial population to see its effects on population size of fish.

Using the first model for the population of fish in a lake ($r_{10000}$ = 1.5, $r_{60000}$ = 1), and a harvest size of 8000, I will investigate the effects of different initial population sizes.

I will now consider initial population sizes of 10000, 20000, 30000, 40000 and 50000.
Initial Population of 10000 with an Annual Harvest of 8000

Instead of using an initial population size of 60000, it is now changed to 10000. The rate of growth is still the same due to the fact that the same model is used. Using excel to get some data, we have:

| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 8000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 10000 | 1.5 |
| 1 | 7000 | 1.53 |
| 2 | 2710 | 1.5729 |
| 3 | -3737.441 | 1.63737441 |

Putting this into a graph, we can see:
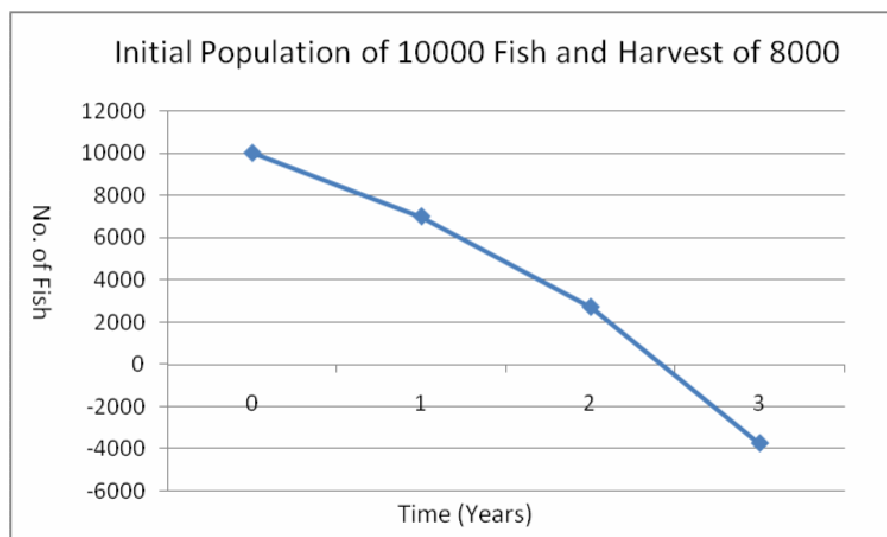
It is clear from the graph that the rate at which the fish population depletes quicker and quicker. This is because a small amount of fish, no matter how large the growth rate, cannot overcome the harvest of 8000. By the third year, the fish population has already depleted. With a harvest of 8000, the government can NOT start an annual harvesting 8000 fish when the population of fish is 10000.

Initial Population of 20000 with an Annual Harvest of 8000

Using Excel to get a data table:

| Year | No. of Fish at beginning of year | Rate |
|------|----------------------------------|------|
| $t$ | $u_{n+1} = u_n \times r_n - 8000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 20000 | 1.4 |
| 1 | 20000 | 1.4 |
| 2 | 20000 | 1.4 |
| 3 | 20000 | 1.4 |

A rather peculiar result appears. At a population of 20000, the growth rate is found to be 1.4, which means that over the next year, the population of fish will increase by $20000 \times 0.4 = 8000$. This is exactly the harvest size by the government, so the population of fish will remain at an exactly stable rate. Therefore it is feasible that the government

can start harvesting when population of fish is 20000. From this result, we can claim that the sustainable limit for a harvest of 8000 fish is 2000 – or can we?

The most vital part of this investigation is to realize that the function for population of fish is still a *quadratic*! In other words, it has 2 solutions (with the exception of a harvest of 9000 when $\Delta = 0$). This can be explained when we solve the quadratic equation for a harvest size of 8000:

$0.00001u_n^2 + 0.6u_n - H \qquad = 0$

$0.00001u_n^2 + 0.6u_n - 8000 \qquad = 0$



We see that there are 2 possible solutions to this quadratic equation: 20000 and 40000. Earlier when I was investigating maximum harvest sizes, I found (but not proved) that the solutions to the quadratic gave the long term sustainable limits for a population of fish.

For a harvest of 8000 fish, we can hypothesise that another sustainable limit is at 40000 fish.

Initial Population of 30000 with an Annual Harvest of 8000

Using Excel to get a data table:

| Year | No. of Fish at beginning of year | Growth Rate |
|------|----------------------------------|-------------|
| $t$ | $u_{n+1} = u_n \times r_n - 8000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 30000 | 1.3 |
| 1 | 31000 | 1.29 |
| 2 | 31990 | 1.2801 |
| 3 | 32950.399 | 1.2705 |
| 4 | 33863.35046 | 1.26137 |
| 5 | 34714.09569 | 1.25286 |
| 6 | 35491.86871 | 1.24508 |
| 7 | 36190.26249 | 1.2381 |
| 8 | 36807.06899 | 1.23193 |

| 9 | 37343.70711 | 1.22656 |
| 10 | 37804.40677 | 1.22196 |
| ………. | …………………….. | …………. |
| 42 | 39997.98901 | 1.20002 |
| 43 | 39998.39117 | 1.20002 |
| 44 | 39998.71291 | 1.20001 |
| 45 | 39998.97031 | 1.20001 |
| 46 | 39999.17624 | 1.20001 |
| 47 | 39999.34098 | 1.20001 |
| 48 | 39999.47278 | 1.20001 |
| 49 | 39999.57822 | 1.2 |
| 50 | 39999.66258 | 1.2 |

In a graph, we can show this as:
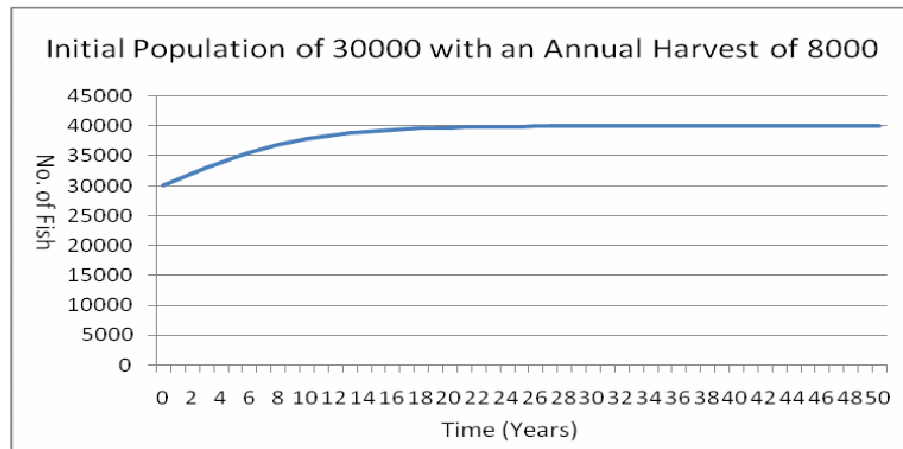


As we can see from this data, it is true that 40000 is also a sustainable limit.

Initial Population of 40000 with an Annual Harvest of 8000

Since we found that 40000 is in fact a sustainable limit, it comes to no surprise that the data table is as follows:

| Year | No. of Fish at beginning of year | Growth Rate |
|---|---|---|
| $t$ | $u_{n+1} = u_n \times r_n - 8000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 40000 | 1.2 |
| 1 | 40000 | 1.2 |
| 2 | 40000 | 1.2 |
| 3 | 40000 | 1.2 |
| 4 | 40000 | 1.2 |

The population of fish will increase by 40000×0.2 = 8000. This is exactly the harvest size by the government, so the population of fish will remain at an exactly stable rate.

Initial Population of 50000 with an Annual Harvest of 8000

Using Excel to get a data table:

| Year | No. of Fish at beginning of year | Rate |
|------|----------------------------------|------|
| $t$ | $u_{n+1} = u_n \times r_n - 8000$ | $r_n = -0.00001 \times u_n + 1.6$ |
| 0 | 50000 | 1.1 |
| 1 | 47000 | 1.13 |
| 2 | 45110 | 1.1489 |
| 3 | 43826.879 | 1.16173121 |
| 4 | 42915.05317 | 1.170849468 |
| 5 | 42247.06719 | 1.177529328 |
| 6 | 41747.16064 | 1.182528394 |
| 7 | 41367.20281 | 1.186327972 |
| 8 | 41075.06981 | 1.189249302 |
| ……… | ……………….. | …………… |
| 33 | 40003.80491 | 1.199961951 |
| 34 | 40003.04378 | 1.199969562 |
| 35 | 40002.43493 | 1.199975651 |
| 36 | 40001.94789 | 1.199980521 |
| 37 | 40001.55827 | 1.199984417 |
| 38 | 40001.24659 | 1.199987534 |
| 39 | 40000.99726 | 1.199990027 |
| 40 | 40000.7978 | 1.199992022 |

Putting this into a graph, we have:
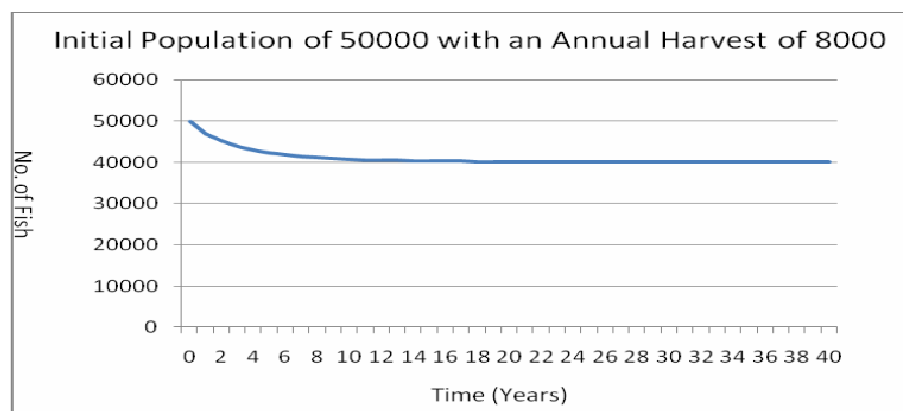
Any initial population of above 40000 will converge to the sustainable limit 40000.
If the harvest begins at any initial population of below 20000, the stock of fish will be depleted over time.

Anything above 20000 will lead the fish to cultivate and reach the sustainable limit of 40000. Only when initial population is 20000 will population stay constant at 20000.

At this stage, one may be thinking 'If growth rates are higher when there is a small population, and lower when there is a high population, why is it possible to have 2 sustainable limits?'

The answer to the question is quite obvious – the rate of growth will be high, however the real increase in no. of fish is dependant on the number of fish present.

| No. of Fish | Growth Rate | Real Increase in Fish |
|:---:|:---:|:---:|
| $u_n$ | $r_n = -0.00001 \times u_n + 1.6$ | $Increase = u_n \times r_n - u_n$ |
| 0 | 1 | 0 |
| 10000 | 1.5 | 5000 |
| 20000 | 1.4 | 8000 |
| 30000 | 1.3 | 9000 |
| 40000 | 1.2 | 8000 |
| 50000 | 1.1 | 5000 |
| 60000 | 1 | 0 |
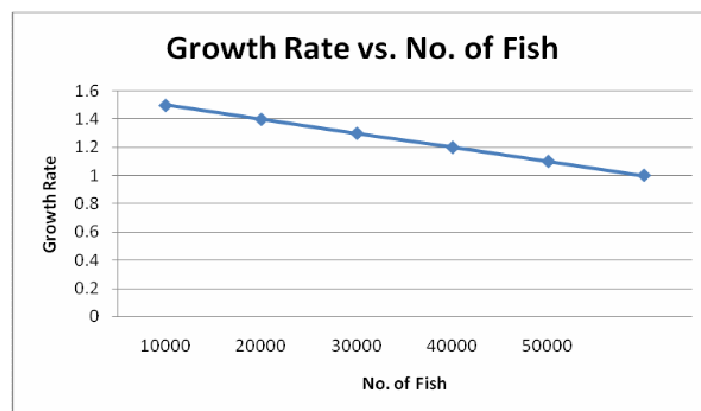


Although it is true that the growth rate of fish will increase as the no. of fish decreases, if we look at the next graph, the real increase in fish is not the same.

**Real Increase in Fish vs. No. of Fish**
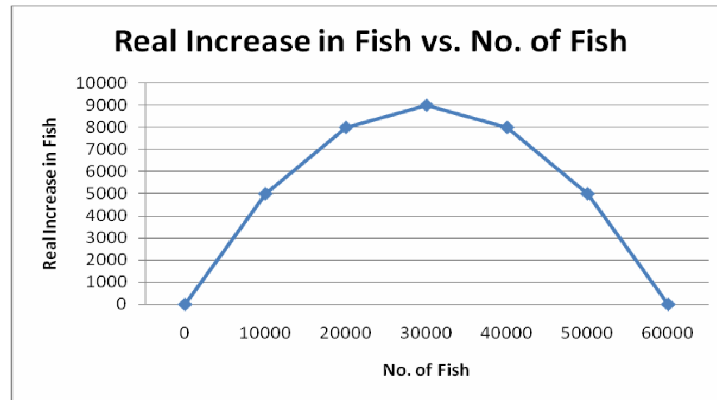
This graph shows that as population increases, the initial increase in fishes goes up, however after a population of 30000 fish, the increase in number of fish begins to slow down. It takes the shape of an inverse parabola.

This helps us explain a lot of things, especially the shape of the graphs throughout this portfolio. Taking the graph for our first model (Initial growth rate of 1.5), we see that the population of fish increases quicker and quicker, however flattens out afterwards until it reaches a limit at 60000.

Using the data that I calculated about real increase in fish, we can now see that the point at which the increase in population size begins to slow down is after population reaches 30000. This coheres with my information about the real growth of fish.



**Population of Fish vs. Time**
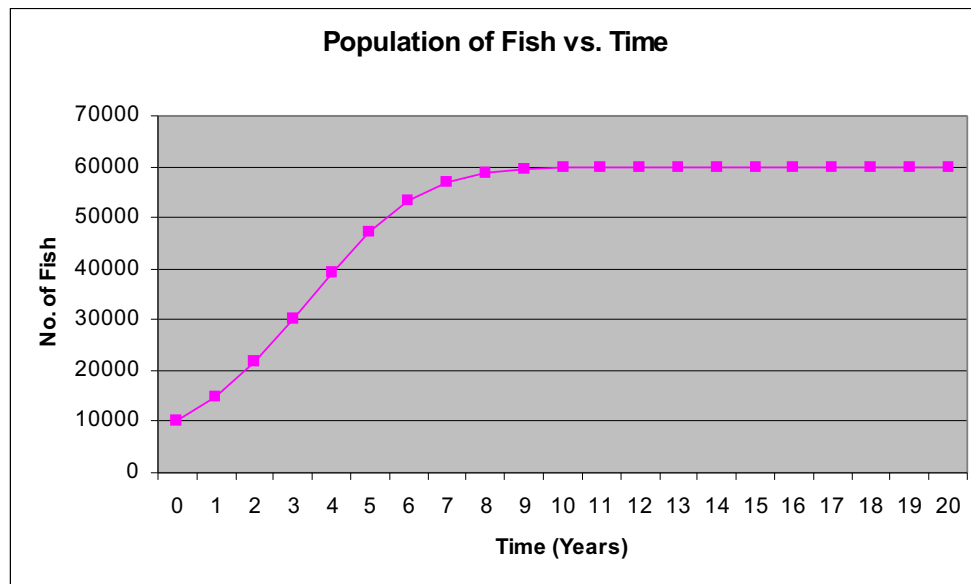
This also helps to explain why it is possible for there to be 2 sustainable limits for one harvest size. Looking back at our table for real increase of fish, we see that there are two points in which a real increase of 8000 fish can be reached. This is at growth rates 1.4 and 1.2; when population size is 20000 or 40000.

From this, we can also say that if there were to be an annual harvest of 5000 fish, the new possible sustainable limits would be 10000 and 50000, as they are the points at which there is an increase of 5000 fish per year, nullifying the harvest.

For a harvest of 9000 however, the only way to allow the real growth rate to reach 9000 is when population is 30000, there is only one solution. So, for a harvest size of 9000, there is only one sustainable limit.

For any harvest size, the way to find out when it is feasible to begin harvesting is by solving the quadratic equation model of the population of fish. The two (or one) values that we obtain for $u_n$ are the sustainable limits. It is only possible to begin harvesting when the population has reached the *lower sustainable limit* i.e. the smaller solution. Any harvest starting before this population is reached will lead to a depletion of fish. Any harvest starting when population is above this will lead to a convergence of the population of fish and the *higher sustainable limit* i.e. the larger solution.